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Approximation of Chirp Functions by Fractional Fourier Series

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Abstract

In this work, we consider the approximation of exponential linear chirps by partial sums of a series called fractional Fourier series on $[-\pi, \pi]$. The chirps belong to an important class of highly oscillatory functions. Such functions are widely used in radar, sonar and communication systems.

The approximation of a function using partial sums of Fourier series is a standard technique in Fourier analysis which decomposes a function into a linear combination of sinusoidal harmonics. However, a major drawback in classical Fourier series is that it does not render itself to the analysis of highly oscillatory functions such as chirps. The approximation by the partial sums of Fourier series is inadequate to match the fast-growing oscillations in chirps. The more oscillations present in the function, the more number of Fourier coefficients are required for the reconstruction.

To overcome this difficulty in an approximation of the exponential linear chirps, we use fractional Fourier series which decomposes a function into a linear combination of chirp harmonics. The fractional Fourier series of exponential linear chirps with an optimum fractional parameter provides the maximum magnitude of fractional Fourier coefficients of small degree which results into the fast decay of fractional Fourier coefficients of large degree. Therefore, a small number of fractional Fourier coefficients are required for the reconstruction of chirps. Moreover, we show monotonicity properties of the fractional Fourier coefficients for a certain parameter range.

Zusammenfassung

In dieser Arbeit betrachten wir die Approximation von exponentiellen linearen Chirps durch so genannte fraktionale Fourier-Reihen auf $[-\pi, \pi]$. Die Chirps gehören zu einer wichtigen Klasse stark oszillierender Funktionen. Solche Funktionen sind weit verbreitet in Radar-, Sonar- und Kommunikationssystemen.

Die Approximation von Funktionen mittels Fourier-Reihen ist eine Standardtechnik der Fourier-Analysis, die eine Funktion in eine Linearkombination von Sinus-Schwingungen zerlegt. Allerdings erweist es sich als Nachteil klassischer Fourier-Reihen, dass sie nicht so gut geeignet sind für die Analyse stark oszillierender Funktionen wie Chirps. Die Approximation durch Fourier-Summen passt daher wenig zu den schnell wachsenden Oszillationen in Chirps. Je mehr Oszillationen in Chirps vorhanden sind, um so mehr Fourier-Koeffizienten sind zur Rekonstruktion nötig.

Um diese Schwierigkeit bei der Approximation exponentieller linearer Chirps zu umgehen, benutzen wir fraktionale Fourier-Reihen, die Funktionen in Linearkombinationen von Chirp-Schwingungen zerlegen. Die fraktionale Fourier-Reihe eines exponentiellen linearen Chirps mit optimalem fraktionalen Parameter hat die betragsmäßig größten Fourier-Koeffizienten mit kleinem Grad und dementsprechend ein schnelles Abklingen der Fourier-Koeffizienten mit großem Grad. Deshalb reicht eine kleine Anzahl von Fourier-Koeffizienten für die Approximation von Chirps. Außerdem werden für gewisse Parameterbereiche Monotonieeigenschaften der fraktionalen Fourier-Koeffizienten gezeigt.

Contents

Chapter 1

Introduction

The decomposition of a function into a suitable basis is a classical problem of applied analysis. Fourier series is a basic example of such an approach. The representation of a function using Fourier series defined on a finite interval $[-\pi, \pi]$ has been extensively studied in the literature. The well-developed theory of Fourier series can be found in the textbooks on Fourier analysis [5, 12, 13, 23, 46] and harmonic analysis [6, 17, 42, 70, 78].

A chirp function (or simply a chirp) is a wave whose instantaneous frequency changes continuously over time. A function that sweeps frequencies linearly over a specific interval of time has the form

$$\mathrm{e}^{\mathrm{i}2\pi(\frac{\mu}{2}t^2+\omega t)}, \quad \mu \in \mathbb{R}\backslash\{0\},\ \omega \in \mathbb{R}. \tag{1.1}$$

The complex function (1.1) is called a chirp and parameter μ is called the chirp rate or the sweep rate [3]. On the contrary to the chirps, a wave function of the form

$$\mathrm{e}^{\mathrm{i}2\pi\omega t} \tag{1.2}$$

contains constant frequency ω. The complex exponentials (1.2) constitute a basis for the space of functions used in the classical Fourier series [46, p. 163].

The English word "chirp" means the sound of a singing bird. Chirps emanate from the Doppler effect, a phenomenon by which the perception of the emitted frequency of a wave is altered as a result of the relative motion of the moving body and the stationary observer. Chirps are often named as Doppler [79] due to this phenomenon. Chirps are found everywhere in nature and used in communication and echolocation of animals. Birds, whales and frogs produce whistles in the form of chirps to communicate with each other. Bats and dolphins use sonar systems to locate food by emitting chirps. Chirps appear in a number of scientific disciplines [21, 59], for instance, physics, mechanics, vibrations, biology and medicine. Chirps have been of great interest in applications [10, 16] in science and technology such as digital communications, bio-medical, geophysical, radar and sonar.

Since chirps are very different functions from the classical functions of mathematics, new methods are required to study such functions. In general, much work has to be done to find the best tools for the analysis of the chirps [21, 32, 59, 61]. Particularly, from mathematical point of view, it is necessary to develop methods that can approximate the chirps efficiently. Fourier series is a commonly used approach for the approximation of a function on a finite interval. A major drawback of the classical Fourier series is that it tries to decompose a function in terms of monochromatic basis functions. Chirps, however, exhibit frequencies that change continuously over time, and the classical Fourier series is unable to detect oscillating patterns present in the chirps. Hence, the classical Fourier series is insufficient regarding approximation and provides slow rate of convergence. These disadvantages led to the development of a potential series called fractional Fourier series (FrFS) [68, 69].

FrFS is a mathematical generalization of the classical Fourier series. It has a potential implementation in every field where the classical Fourier series is applied. It possesses elegant properties and has applications [1, 9, 69] in science, engineering and technology. In fact, FrFS enhances the efficiency of the classical Fourier series by using a more general parameterized orthonormal basis with an adjustable parameter to expand a chirp function into a linear combination of such a basis. The parameterized orthonormal basis used for the decomposition of a chirp function has the ability to reconstruct the chirp functions efficiently. These chirp basis will be more convenient and efficient which makes FrFS a robust tool for the analysis of chirps. The partial sums of FrFS provide a very good approximation using a few fractional Fourier coefficients. Thus, we have a new series and a new basis which can achieve better performance than the classical Fourier series.

In short, we replace the standard Fourier basis on $[-\pi, \pi]$ given by

$$\left\{\varphi_k(t) = \mathrm{e}^{\mathrm{i}kt} : k \in \mathbb{Z}\right\} \tag{1.3}$$

with the basis containing the chirp functions

$$\left\{\phi_{k,\alpha}(t) = \mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{\mathrm{i}kt} : k \in \mathbb{Z}\right\} \tag{1.4}$$

for $0 < \alpha \leq \frac{\pi}{2}$. The decomposition of a function into a linear combination of Fourier basis functions (1.3) is given by

$$\sum_{k\in\mathbb{Z}} \hat{f}(k)\varphi_k(t),$$

with Fourier coefficients

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{\varphi_k(t)}\,\mathrm{d}t$$

is now replaced by the decomposition of a function into a linear combination of chirp basis functions (1.4) having the fixed chirp rate α

$$\sum_{k\in\mathbb{Z}} \hat{f}_\alpha(k)\phi_{k,\alpha}(t),$$

with fractional Fourier coefficients

$$\hat{f}_\alpha(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{\phi_{k,\alpha}(t)}\ \mathrm{d}t \tag{1.5}$$

for $0 < \alpha \leq \frac{\pi}{2}$. The investigation of the basis (1.4) for the approximation of linear chirps is the primary aim of this thesis.

Remark. The terminology "fractional" used in fractional Fourier basis has nothing in common with fractional calculus, i.e., with fractional order derivatives and fractional order integrals. The term "fractional" originated from fractional Fourier transforms where an equivalent representation of

$$\alpha = r\frac{\pi}{2}, \quad 0 < r \leq 1$$

is often considered. For each value of $0 < r \leq 1$, the parameter α can be represented as a fractional multiple of $\frac{\pi}{2}$.

The fractional Fourier coefficients are integrals of the form (1.5) which involve quadratic exponentials. Therefore, the FrFS representation of chirps on a finite interval is limited to the numerical experiments [14, 30, 68, 69]. Moreover, the integrals (1.5) can be expressed in terms of error functions of a complex variable. Since the error functions of a complex variable are special functions of mathematical physics with no closed form available, it is more complicated to obtain mathematical proofs with the error functions of a complex variable. That's why the mathematical formulation of FrFS on a finite interval has been less considered so far.

We prove that the approximation of exponential linear chirps by partial sums of FrFS with an optimal parameter has an advantage over the classical Fourier series. We attempt to establish a connection between FrFS concepts from numerical analysis and mathematical analysis. The results of this thesis will bridge the gap between applications and well-developed mathematical theory of FrFS.

To sum up, the intention of this thesis is twofold: First, to develop a mathematical theory of FrFS for the space $L^2[-\pi, \pi]$, particularly to study FrFS in the fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$ and prove that the performance of FrFS with an optimal parameter gives the best results in chirp approximation, and second, to derive the properties of fractional Fourier coefficients of exponential linear chirps such as fractional Fourier coefficients of zero degree, fractional Fourier coefficients of large degree and L^2 error in FrFS expansion of the exponential linear chirps.

1.1 Existing Literature

The classical Fourier transform has been used for applications in signal analysis and optics for more than hundred years. Recently, it was found the classical Fourier transform is not efficient for solving certain mathematical problems and its physical applications. To address deficiency in the classical Fourier transform, fractional Fourier transform has been introduced. The fractional Fourier transform is a mathematical generalization of the classical Fourier transform with a parameter α. This transform has ability to solve mathematical problems in a more effective way than the classical Fourier transform. Later on, the concept of fractional Fourier transform has been employed in solving mathematical problems of signal processing and optics [18, 36, 37].

The prehistory of fractional Fourier analysis starts by the publication of Wiener [57] in 1929 to solve certain ordinary differential equations and partial differential equations of quantum mechanics. Namias [77] in 1980, unaware of Wiener's work, reinvented the fractionalization in Fourier transform to solve ordinary differential equations and partial differential equations emerging in quantum mechanics from classical quadratic Hamiltonian. He regarded the classic Fourier transform as a transform of order one and identity transform of order zero. Later on McBride [4] developed mathematical theory of the fractional Fourier transform by establishing its its algebra and calculus. Kerr [26] provided a mathematical framework for the theory of fractional Fourier transform on the space $L^2(\mathbb{R})$ and discussed its applications to partial differential equations. In 1994, Almeida [45] considered the fractional Fourier transform as a rotation of the time-frequency plane.

The research on fractional Fourier transform and its applications [18, 37, 38, 75] gained momentum in 1990's, and it is still proceeding. A cutting edge textbook on fractional Fourier transform written by Ozaktas is [36]. The fast computation techniques of the discrete fractional Fourier transform are given in [22, 35, 63]. A lot of research papers on fractional Fourier transform and its practical applications are available in the literature [3, 24, 66, 67].

In comparison to the fractional Fourier transform, very little literature currently exists about FrFS expansions of chirp functions. There are a few research papers that show recent attempts on FrFS and its applications. Pei et al. [69] introduced fractional Fourier basis and FrFS expansion of linear chirps. The generalization of FrFS to the chirps with an arbitrary central frequency was pursued by Barkat and Yingtuo [14]. They proposed shifted fractional Fourier basis under the topic of modified FrFS. Yu [30] performed numerical simulations of Gaussian chirp signals by using FrFS expansions. The analysis of chirped pulses with fractional order Fourier series was given by Cöetmellec [68], and Brunel [50, 51] did fractional order Fourier series analysis of ultra-short pulse characterization and SPIDER interferograms.

The Gibbs phenomenon and its resolution in one dimension using the inverse method was presented by Zhu et al. [39, 40] whereas Ding and Zhu [52] discussed the two-dimensional Gibbs phenomenon and its resolution. The relation between FrFS and fractional Fourier transform was established by Candan [15].

1.2 Outline and the Results

We provide an outline of this thesis and a short overview of the main results. The detailed information is given at the start of the respective chapter.

Chapter 2 provides an introduction to chirps and exponential linear chirps. We recall some of the well-known properties of the orthonormal basis of the Hilbert space $L^2[-\pi,\pi]$ and completeness of the Fourier system. We introduce FrFS for the space $L^2[-\pi,\pi]$ of square integrable functions. Key results include completeness of the fractional Fourier system and uniform convergence of the FrFS.

In Chapter 3, we derive an estimate for the error function on the complex plane including a method to split the error function into real and imaginary part. We provide the asymptotic expansion of generalized error functions and derive an estimate of the remainder term.

The main chapter of the thesis is Chapter 4 where we prove new inequalities related to fractional Fourier coefficients of exponential linear chirps. We prove rigorously that by adjusting the fractional Fourier parameter to a certain value, we obtain optimal FrFS which approximates the linear chirps efficiently.

We describe the problem of finding the optimal fractional Fourier parameter as follows: The approximation of a chirp function of the form

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi,\pi], \tag{1.6}$$

defined for all $\gamma > 0$ and $\mu \in \mathbb{R}^+$, using the partial sums of classical Fourier series provides all Fourier coefficients of small magnitude. Therefore, a large number of Fourier coefficients are required to reconstruct the chirp (1.6). On the other hand, when the chirp function (1.6) is approximated by partial sums of FrFS with a certain fractional Fourier parameter, fractional Fourier coefficient of zero degree has the maximum value and fractional Fourier coefficients of large degree rapidly go to zero. Consequently, a small number of fractional Fourier coefficients need to be computed to obtain an accurate approximation of the chirp (1.6). The optimized fractional Fourier parameter occurs when oscillating pattern of the chirp (1.6), i.e., $\pi\mu$ matches with the oscillating pattern of the fractional Fourier basis functions (1.4), i.e., $\frac{1}{2}\cot\alpha$. More precisely, we write the condition

$$\frac{1}{2}\cot\alpha = \pi\mu,$$

which leads to the following relation

$$\alpha^* = \arctan\left(\frac{1}{2\pi\mu}\right). \tag{1.7}$$

The optimal parameter (1.7) can be found for every value μ. An important property of fractional Fourier coefficients with the optimal fractional Fourier parameter is the faster decay of the fractional Fourier coefficients for the chirp functions. The benefit of using such expansions to approximate a chirp function is the better convergence of FrFS compared to the classical Fourier series. Moreover, the convergence is uniform on $[-\pi, \pi]$ and no Gibbs phenomenon on the boundary of the interval is observed.

In this chapter, we propose an approximation of exponential linear chirps by the partial sums of FrFS. To show the benefits of the proposed approximation, we derive new inequalities concerning fractional Fourier coefficients of the exponential linear chirps and its relationship with the optimal parameter α^*. We describe the proved results as follows: We prove the maximality of the absolute value of the fractional Fourier coefficients $\left|\hat{f}_\alpha(0)\right|$ with the optimal fractional Fourier parameter α^* in the following theorem.

Theorem 1 *Let*

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ *and* $\mu \in \mathbb{R}^+$. *Then for all* α *with* $0 < \alpha \leq \frac{\pi}{2}$ *and* $\alpha \neq \alpha^*$, *we have*

$$\left|\hat{f}_{\alpha^*}(0)\right| > \left|\hat{f}_\alpha(0)\right|.$$

We prove the minimality of the absolute value of the fractional Fourier coefficients $\left|\hat{f}_\alpha(k)\right|$ with the optimal fractional Fourier parameter α^* in the following theorem.

Theorem 2 *Let*

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ *and* $\mu \in \mathbb{R}^+$. *There exists* $N(\gamma, \mu) \in \mathbb{N}_0$ *such that for all* $|k| > N(\gamma, \mu)$, *we have*

$$\left|\hat{f}_{\alpha^*}(k)\right| < \left|\hat{f}_\alpha(k)\right|$$

on $0 < \alpha \leq \frac{\pi}{2}$ *when* $\alpha \neq \alpha^*$, *where* α^* *is the optimal fractional Fourier parameter given in* (1.7).

We consider the topic of monotonicity of fractional Fourier coefficients over a domain $\mathcal{D}$ given by

$$\mathcal{D} = \left\{(\gamma,\mu) : \gamma \geq \frac{1}{2\pi},\ 0 < \mu \leq \frac{\sqrt{\gamma}}{8\pi}\left(\lambda e^{\pi^3\gamma} + \sqrt{(\lambda e^{\pi^3\gamma})^2 - 64\pi^2\gamma}\right)\right\}, \tag{1.8}$$

where

$$\lambda = \left(1 - \frac{2\sqrt{2}e^{\frac{-\pi^2}{2}}}{\pi^{\frac{3}{2}}}\right)^2.$$

We prove that for all $(\gamma,\mu) \in \mathcal{D}$, fractional Fourier coefficients of zero degree and fractional Fourier coefficients of large degree are monotone. The domain $\mathcal{D}$ is illustrated in Figure 1.1.

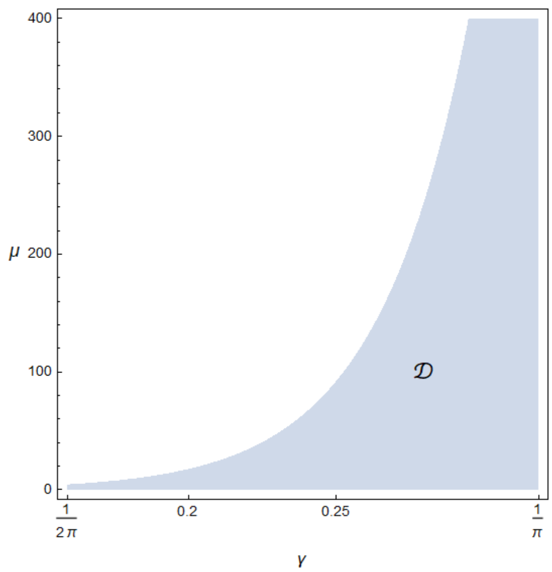


Figure 1.1: The region plot of the domain $\mathcal{D}$ for $\frac{1}{2\pi} \leq \gamma \leq \frac{1}{\pi}$ and $0 < \mu \leq 400$.

The following theorem proves that for all $(\gamma,\mu) \in \mathcal{D}$, the fractional Fourier coefficients $\left|\hat{f}_\alpha(0)\right|$ are monotone decreasing on $\alpha > \alpha^*$ and monotone increasing on $\alpha_1 \leq \alpha < \alpha^*$, where $0 < \alpha_1 < \alpha^*$.

Theorem 3 *Let*

$$f(t) = f_\gamma^\mu(t) = e^{-\pi(\gamma+i\mu)t^2}, \quad t \in [-\pi,\pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ and $\mu \in \mathbb{R}^+$. Let $(\gamma,\mu) \in \mathcal{D}$ given in (1.8). Then there exists $\alpha_1 = \alpha_1(\gamma,\mu)$ such that

$$\left|\hat{f}_{\alpha_R}(0)\right| > \left|\hat{f}_{\beta_R}(0)\right| \quad \textit{for} \quad \alpha^* < \alpha_R < \beta_R \leq \frac{\pi}{2}$$

and

$$\left|\hat{f}_{\beta_L}(0)\right| < \left|\hat{f}_{\alpha_L}(0)\right| \quad \textit{for} \quad \alpha_1 \leq \beta_L < \alpha_L < \alpha^*,$$

where $0 < \alpha_1 < \alpha^*$ *and* α^* *is the optimal fractional Fourier parameter given in* (1.7).

The following theorem proves that for all $(\gamma, \mu) \in \mathcal{D}$, the fractional Fourier coefficients $\left|\hat{f}_\alpha(k)\right|$ for $|k| > N(\gamma, \mu)$ are monotone increasing on $\alpha > \alpha^*$ and monotone decreasing on $\alpha_1 \leq \alpha < \alpha^*$, where $0 < \alpha_1 < \alpha^*$.

Theorem 4 *Let*

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma + \mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ *and* $\mu \in \mathbb{R}^+$. *Let* $(\gamma, \mu) \in \mathcal{D}$ *given in* (1.8). *Then there exists* $N(\gamma, \mu) \in \mathbb{N}_0$ *and* $\alpha_1 = \alpha_1(\gamma, \mu)$ *such that for all* $|k| > N(\gamma, \mu)$

$$\left|\hat{f}_{\alpha_R}(k)\right| < \left|\hat{f}_{\beta_R}(k)\right| \quad \text{for} \quad \alpha^* < \alpha_R < \beta_R \leq \frac{\pi}{2}$$

and

$$\left|\hat{f}_{\beta_L}(k)\right| > \left|\hat{f}_{\alpha_L}(k)\right| \quad \text{for} \quad \alpha_1 \leq \beta_L < \alpha_L < \alpha^*,$$

with $0 < \alpha_1 < \alpha^*$, *where* α^* *is the optimal fractional Fourier parameter given in* (1.7).

Finally, we draw conclusions and outline directions for future research. Throughout the whole thesis, we use Mathematica 11.3 to plot graphs and perform numerical calculations, especially the ListContourPlot with the monotonicity domain $\mathcal{D}$.

Remark. When considering the results on the domain $\mathcal{D}$, Theorems 1 and 2 are corollaries of Theorems 3 and 4 respectively.

Chapter 2

Chirp Functions and Fractional Fourier Series on $L^2[-\pi, \pi]$

To give the essential background of this thesis, we provide a mathematical overview of chirp functions and FrFS for the space $L^2[-\pi, \pi]$ of square integrable functions. We start with the mathematical introduction to the chirps and exponential linear chirps in Section 2.1. We present an overview of the orthonormal basis in Hilbert spaces $L^2[-\pi, \pi]$ and a well-known theorem on completeness of the Fourier system in Section 2.2. We provide an introduction to fractional Fourier analysis and fractional Fourier system in Section 2.3. This section also addresses the orthogonality and completeness of the fractional Fourier system. Finally, we define FrFS for the space $L^2[-\pi, \pi]$ of square integrable functions and prove its uniform convergence.

2.1 Chirps

A chirp function [2, 60] is a function that sweeps all frequencies through a specific interval while it progresses in time. Mathematically, the functions which are highly oscillatory are commonly known as chirps.

Definition 2.1.1. Chirps are the functions of the form

$$f(t) = A(t)\mathrm{e}^{\mathrm{i}p(t)}, \quad t \in [-\pi, \pi], \tag{2.1}$$

where $A(t) > 0$ for all $t \in [-\pi, \pi]$ is an amplitude function and $p(t) \in \mathbb{R}$ is a phase function. The phase $p(t)$ has fast oscillations and the amplitude function $A(t)$ is a smooth function with slow variations as compared to the phase $p(t)$.

In Definition 2.1.1, the slow variation conditions [21, 59, 71] need to be more precise by the following inequalities

$$\left|\frac{A'(t)}{A(t)p'(t)}\right| \ll 1 \quad \text{and} \quad \frac{|p''(t)|}{\left(p'(t)\right)^2} \ll 1, \quad A(t) > 0,\ p'(t) \neq 0.$$

Definition 2.1.2. The instantaneous frequency of the chirp f is defined by

$$\omega(t) = \frac{1}{2\pi} p'(t). \tag{2.2}$$

When a chirp oscillates with the instantaneous frequency $\omega(t) > 0$, then the chirp is called up-chirp. In case $\omega(t) < 0$, the chirp is down-chirp. If one of the functions $A(t)$ or $\omega(t)$ is not constant, then the chirp is a non-stationary complex sinusoid [19]. Moreover, a chirp is an amplitude modulated chirp when the amplitude function $A(t)$ to be non-constant whereas the chirp is a frequency modulated chirp when the instantaneous frequency $\omega(t)$ to be non-constant. We consider the chirps which are both amplitude modulated and frequency modulated, i.e., both $A(t)$ and $\omega(t)$ to be the functions of time t.

2.1.1 Linear Chirps

There can be many types of chirps such as linear chirps, quadratic chirps, hyperbolic chirps and trigonometric chirps. A detailed discussion of the mathematical concepts of the chirps can found in [20, 72, 80]. We are mainly interested in linear chirps that can be defined as follows:

Definition 2.1.3. A chirp of the form (2.1) is called a linear chirp if the phase function $p(t)$ is a quadratic function

$$p(t) = 2\pi \left(\frac{\mu_2}{2} t^2 + \mu_1 t + \mu_0 \right). \tag{2.3}$$

In (2.3), the parameter μ_2 is called the chirp rate, μ_1 represents central frequency and μ_0 is phase offset defined at the temporal center $t = 0$ of a pulse of length 2π. Applying (2.2) to (2.3), the instantaneous frequency swept linearly in time is given by

$$\omega(t) = \mu_2 t + \mu_1.$$

2.1.2 Exponential Linear Chirps

Since the human perception of sound intensity is recognized as a logarithmic function, so $A(t)$ is taken as a real exponential function which produces a compact representation of the linear chirp [74, p. 4]. For exponential linear chirps, we replace $A(t)$ by an exponential function of the form

$$A(t) = \mathrm{e}^{-\pi\gamma t^2}, \quad \gamma > 0,$$

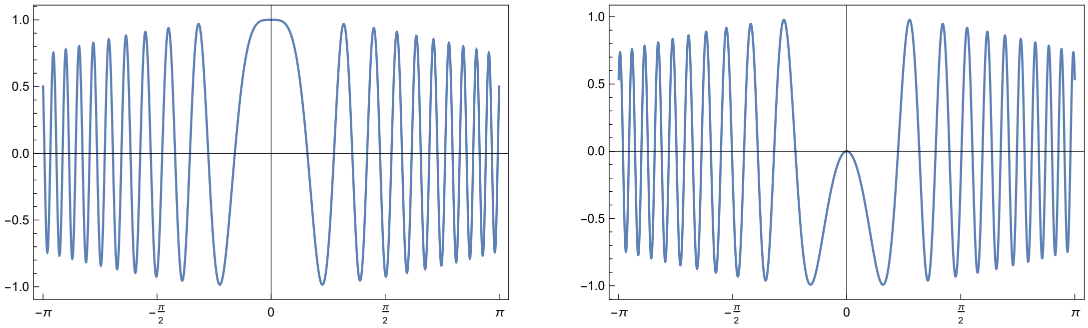
and $p(t)$ by leading term of quadratic polynomial of the form

$$p(t) = -\pi\mu t^2, \quad \mu \in \mathbb{R}\backslash\{0\},$$

then the chirp (2.1) is called an exponential linear chirp, denoted by $f_\gamma^\mu(t)$, given by

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma + \mathrm{i}\mu)t^2}. \tag{2.4}$$

Chirps have been extensively studied on the real line in the context of the fractional Fourier transforms [11, 32, 61, 62, 67]. The analysis of chirps over a finite interval, that is, to chirps over $[-\pi,\pi]$ has been less considered. In this thesis, we study chirps of the form (2.4) over the finite interval $[-\pi,\pi]$ in the context of FrFS. The graph of an exponential linear chirp is shown in Figure 2.1.



(a) (b)

Figure 2.1: For $\gamma = 0.01$ and $\mu = 2$, the graph of chirp function $f_\gamma^\mu(t)$ on $-\pi \leq t \leq \pi$; (a) the real part, and (b) the imaginary part.

2.2 Orthonormal Basis in Hilbert Space

The space we consider is the Hilbert space $L^2[-\pi,\pi]$ of square integrable complex-valued functions f that satisfy

$$\int_{-\pi}^{\pi} |f(t)|^2 \,\mathrm{d}t < \infty.$$

For two complex valued functions f and g in $L^2[-\pi,\pi]$, we define the inner product between f and g to be

$$\langle f, g\rangle_{L^2[-\pi,\pi]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}t, \tag{2.5}$$

where $\overline{g(t)}$ is the complex conjugate of $g(t)$. Moreover, the norm of f to be associated is

$$\|f\|_{L^2[-\pi,\pi]} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} = \langle f, f\rangle^{\frac{1}{2}}.$$

An essential system of functions is the system of harmonic complex exponentials given by

$$\left\{\varphi_k(t) = \mathrm{e}^{\mathrm{i}kt} : k \in \mathbb{Z}\right\}. \tag{2.6}$$

Theorem 2.2.1. *The system of functions* (2.6) *forms an orthonormal basis for the Hilbert space* $L^2[-\pi,\pi]$.

Proof. The proof is a standard result in Fourier analysis [65, p. 8]. ■

2.3 Fractional Fourier Analysis

This section provides a brief introduction to FrFS for the Hilbert space $L^2[-\pi,\pi]$. First, we define fractional Fourier system and prove its completeness. Then we give an overview of FrFS, and show limiting behavior of fractional Fourier coefficients. In the last part, we prove uniform convergence of FrFS.

2.3.1 Fractional Fourier System

Now we discuss a vital system, called fractional Fourier system, which deals with mixed time and frequency components. In fractional Fourier analysis, a function is decomposed into a linear combination of chirp basis functions with a fixed sweep rate determined by α, where $0<\alpha\leq\frac{\pi}{2}$. The domain $0<\alpha\leq\frac{\pi}{2}$ is called fractional Fourier parameter domain. The chirps play the same role in fractional Fourier analysis as sinusoids in Fourier analysis.

Definition 2.3.1. Let $0<\alpha\leq\frac{\pi}{2}$ be the fractional Fourier parameter domain. A collection of functions

$$\{\phi_{k,\alpha}(t):k\in\mathbb{Z}\} \tag{2.7}$$

is said to be a fractional Fourier system for the space $L^2[-\pi,\pi]$, where each function $\phi_{k,\alpha}(t)$ is given by

$$\phi_{k,\alpha}(t)=\mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{\mathrm{i}kt},\quad t\in[-\pi,\pi].$$

The system of functions (2.7) forms an orthonormal basis for the space $L^2[-\pi,\pi]$. We prove the following theorem.

Theorem 2.3.1. *Let $0<\alpha\leq\frac{\pi}{2}$ be the fractional Fourier parameter domain. For all $k\in\mathbb{Z}$, the complex valued functions*

$$\phi_{k,\alpha}(t)=\mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{\mathrm{i}kt},\ t\in[-\pi,\pi]$$

constitute an orthonormal basis for $L^2[-\pi,\pi]$, which we will call fractional Fourier basis.

Proof. The orthogonality of the functions $\phi_{k,\alpha}(t)$ can be proved by simple calculations using the inner product defined in (2.5)

$$\langle\phi_{l,\alpha},\phi_{m,\alpha}\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_{l,\alpha}(t)\overline{\phi_{m,\alpha}(t)}\,\mathrm{d}t. \tag{2.8}$$

Substituting $\phi_{l,\alpha}(t)$ and $\overline{\phi_{m,\alpha}(t)}$ into (2.8), we have

$$\begin{aligned}\langle\phi_{l,\alpha},\phi_{m,\alpha}\rangle&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{\mathrm{i}lt}\mathrm{e}^{\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{-\mathrm{i}mt}\,\mathrm{d}t\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(l-m)t}\,\mathrm{d}t=\delta_{l,m},\quad l,\ m\in\mathbb{N}_0,\end{aligned}$$

where $\delta_{l,m}$ is Kronecker delta function

$$\delta_{l,m} = \begin{cases} 1, & \text{for } l = m, \\ 0, & \text{for } l \neq m. \end{cases}$$

Therefore, the system $\{\phi_{k,\alpha}(t) : k \in \mathbb{Z}\}$ is an orthonormal system in $L^2[-\pi, \pi]$. Its still remains to prove the completeness of the system $\{\phi_{k,\alpha}(t) : k \in \mathbb{Z}\}$. For $f \in L^2[-\pi, \pi]$, we shall show that $\langle f, \phi_{k,\alpha}\rangle = 0$ for all $k \in \mathbb{Z}$ implies $f = 0$ a.e.

Suppose $\langle f, \phi_{k,\alpha}\rangle = 0$ for all $k \in \mathbb{Z}$. Then

$$\langle f, \phi_{k,\alpha}\rangle = \left\langle f, \mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{\mathrm{i}kt}\right\rangle = \left\langle f\mathrm{e}^{\mathrm{i}\frac{t^2}{2}\cot\alpha}, \varphi_k\right\rangle = \langle g, \varphi_k\rangle,$$

where

$$g(t) = f(t)\mathrm{e}^{\mathrm{i}\frac{t^2}{2}\cot\alpha} \tag{2.9}$$

and

$$\varphi_k(t) = \mathrm{e}^{\mathrm{i}kt}, \quad k \in \mathbb{Z}$$

constitute a basis for $L^2[-\pi, \pi]$. Since the Fourier system $\{\varphi_k\}_{k\in\mathbb{Z}}$ is complete, we write $g = 0$ in $L^2[-\pi, \pi]$ or equivalently $g = 0$ a.e. From (2.9), we conclude $f = 0$ a.e. ■

The fractional Fourier basis is a generalization of Fourier basis and has some essential properties. The parameter α in $\phi_{k,\alpha}(t)$ is called fractional Fourier parameter with the fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$. One of the most interesting properties of the fractional Fourier basis functions is the adjustment of the fractional Fourier parameter α to some suitable value. By selecting an appropriate value of the fractional Fourier parameter α, the functions in fractional Fourier system are sinusoids or impulse functions. Figure 2.2 illustrates the fractional Fourier basis functions.

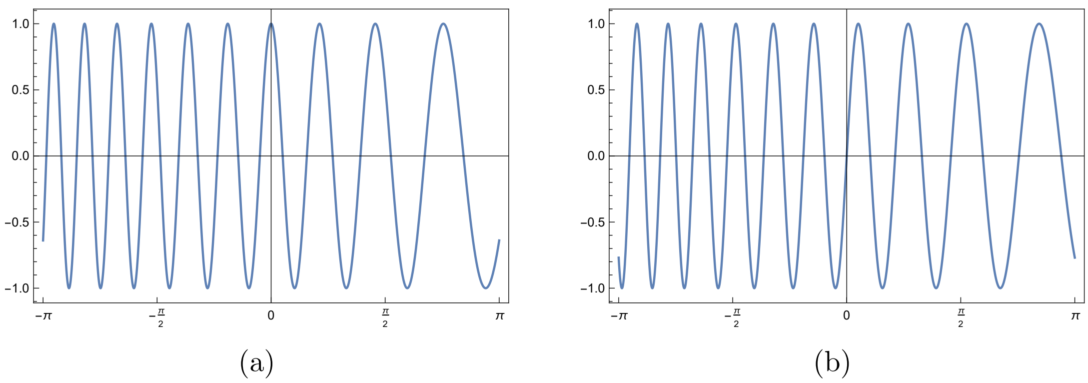


Figure 2.2: For $k = 10$ and $\alpha = \frac{\pi}{6}$, the graph of chirp basis function $\phi_{k,\alpha}(t)$ on $-\pi \leq t \leq \pi$; (a) the real part, and (b) the imaginary part.

2.3.2 Fractional Fourier Series

In comparison to the classical Fourier series, there is a possibility to develop a series called fractional Fourier series (FrFS) [50, 68, 69]. The FrFS is an important extension to the classical Fourier series. We introduce the FrFS for the space $L^2[-\pi,\pi]$ of square integrable functions. Let $0 < \alpha \leq \frac{\pi}{2}$ be the fractional Fourier parameter domain. Consider a function $f \in L^2[-\pi,\pi]$ and the fractional Fourier basis

$$\{\phi_{k,\alpha}(t) : k \in \mathbb{Z}\} \tag{2.10}$$

on $[-\pi,\pi]$, then the fractional Fourier coefficients

$$\left\{\hat{f}_{k,\alpha}(t) : k \in \mathbb{Z}\right\}$$

of f with respect to (2.10) are defined by

$$\hat{f}_\alpha(k) = \langle f, \phi_{k,\alpha}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t,$$

where $\langle .,. \rangle$ is the inner product defined in (2.5). The FrFS of f with respect to basis (2.10) is the series given by

$$\mathcal{F}_\alpha(t) = \sum_{k\in\mathbb{Z}} \hat{f}_\alpha(k)\phi_{k,\alpha}(t). \tag{2.11}$$

The N-th partial sum, $S_{N,\alpha}(t)$, of the fractional Fourier series is

$$S_{N,\alpha}(t) = \sum_{|k|\leq N} \hat{f}_\alpha(k)\phi_{k,\alpha}(t), \quad N \in \mathbb{N}_0.$$

The fractional Fourier coefficients $\hat{f}_\alpha(k)$ are orthogonal projections of f onto fractional Fourier space spanned by $\phi_{k,\alpha}(t)$ for $|k| \leq N$. Therefore, $S_{N,\alpha}$ is the orthogonal projection of f onto the fractional Fourier space of order N

$$F_{N,\alpha} = \operatorname{span}\{\phi_{k,\alpha}(t) : |k| \leq N\}.$$

To approximate a function by the partial sums of FrFS, the fractional Fourier coefficients $\hat{f}_\alpha(k)$ can be computed for each value of α in the fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$. In practice, selection of the fractional Fourier parameter α has a number of consequences for the FrFS defined in (2.11). For instance, in case of exponential linear chirps, there is one specific fractional Fourier parameter for which the series converges rapidly. Thus, the process of decomposition with such a parameter will be of particular interest in chirp analysis. The investigation of a fractional Fourier parameter which can reconstruct the chirp functions with the least error is an emerging problem. There have been some efforts to search the optimal parameter [16, 68, 69]. We discuss this topic in detail in Chapter 4.

Remark. The FrFS can be regarded as a family of series $\{\mathcal{F}_\alpha\}$ indexed by a parameter α with $0 < \alpha \leq \frac{\pi}{2}$ such that $\mathcal{F}_{\frac{\pi}{2}} = \mathcal{F}(t)$ is the classical Fourier series.

2.3.3 Limiting Value of Fractional Fourier Coefficients

Now we examine the case $\alpha \to 0^+$ to gain more insight into the limiting behavior of fractional Fourier coefficients. We prove the following theorem.

Theorem 2.3.2. *Let $0 < \alpha \le \frac{\pi}{2}$ be the fractional Fourier parameter domain. For $f \in L^2[-\pi,\pi]$ and $k \in \mathbb{Z}$, we have*

$$\lim_{\alpha\to 0^+} \hat{f}_\alpha(k) = 0.$$

Proof. The fractional Fourier coefficients of a function $f \in L^2[-\pi,\pi]$ are given by

$$\begin{aligned}\hat{f}_\alpha(k) &= \frac{1}{2\pi}\int_{-\pi}^{0} f(t)\mathrm{e}^{\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t + \frac{1}{2\pi}\int_{0}^{\pi} f(t)\mathrm{e}^{\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t\\ &= I_{-\pi}(\alpha) + I_\pi(\alpha).\end{aligned}$$

Let us consider the integral

$$I_\pi(\alpha) = \frac{1}{2\pi}\int_0^\pi f(t)\mathrm{e}^{\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t. \tag{2.12}$$

Let $g_k(t) = f(t)\mathrm{e}^{-\mathrm{i}kt}$, where $k \in \mathbb{Z}$ and $\beta = \frac{1}{2}\cot\alpha$. Then the integral (2.12) becomes

$$I_\pi(\beta) = \frac{1}{2\pi}\int_0^\pi g_k(t)\mathrm{e}^{\mathrm{i}\beta t^2}\,\mathrm{d}t. \tag{2.13}$$

We use substitution method to simplify the integral (2.13). We set $t^2 = u$ which gives $\mathrm{d}t = \frac{\mathrm{d}u}{2\sqrt{u}}$. The integral (2.13) becomes

$$I_\pi(\beta) = \frac{1}{2\pi}\int_0^{\pi^2} \frac{g_k(\sqrt{u})}{2\sqrt{u}}\mathrm{e}^{\mathrm{i}\beta u}\,\mathrm{d}u.$$

For each $k \in \mathbb{Z}$, the function $\frac{g_k(\sqrt{u})}{2\sqrt{u}} \in L^1[0,\pi^2]$. Therefore, by Riemann- Lebesgue Lemma [49, p. 18], the integral $I_\pi(\beta) \to 0$ as $\beta \to \infty$ or the integral $I_\pi(\alpha) \to 0$ as $\alpha \to 0^+$. Similarly, we can show $I_{-\pi}(\alpha) \to 0$ as $\alpha \to 0^+$. Hence

$$\lim_{\alpha\to 0^+} \hat{f}_\alpha(k) = 0.$$

■

Theorem 2.3.2 shows that fractional Fourier coefficients exist only within the fractional Fourier parameter domain $0 < \alpha \le \frac{\pi}{2}$.

2.3.4 Uniform Convergence of Fractional Fourier Series

The completeness of the fractional Fourier system implies that for a continuous function f on $[-\pi, \pi]$, the FrFS for f converges to f in the mean square sense. For given α with $0 < \alpha \leq \frac{\pi}{2}$ and $\varepsilon > 0$, there exists $N(\varepsilon, \alpha)$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - S_{N,\alpha}(x)|^2 \, \mathrm{d}x < \varepsilon \quad \text{for all } n \geq N,$$

and Parseval's equation holds

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x = \sum_{k \in \mathbb{Z}} \left| \hat{f}_\alpha(k) \right|^2.$$

The mean square error is defined as

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - S_{N,\alpha}(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = \|f - S_{N,\alpha}\|_2 = \left(\sum_{|k|>N} \left| \hat{f}_\alpha(k) \right|^2 \right)^{\frac{1}{2}}.$$

We discuss the uniform convergence of the FrFS. The most straightforward criterion for uniform convergence of the functional series is the Weierstrass theorem. We prove the uniform convergence of the FrFS in the following theorem.

Theorem 2.3.3. *Let $0 < \alpha \leq \frac{\pi}{2}$ be the fractional Fourier parameter domain. Let f be a continuous function on the interval $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$ and that f' is piece-wise continuous. Then N-th partial sum of the FrFS of f, converges uniformly on $[-\pi, \pi]$, i.e.,*

$$\max_{-\pi \leq t \leq \pi} |f(t) - S_{N,\alpha}(t)| \to 0 \quad \text{as} \quad N \to \infty.$$

Proof. First, we establish a relation between the FrFS and the classical Fourier series of a continuous function f. Let $\mathcal{F}_\alpha$ be the FrFS representation of a function f given by

$$\mathcal{F}_\alpha(t) = \sum_{k \in \mathbb{Z}} \hat{f}_\alpha(k) \phi_{k,\alpha}(t), \tag{2.14}$$

where the fractional Fourier coefficients are

$$\hat{f}_\alpha(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) \mathrm{e}^{\mathrm{i}\frac{u^2}{2}\cot\alpha} \mathrm{e}^{-\mathrm{i}ku} \, \mathrm{d}u. \tag{2.15}$$

Substituting (2.15) into (2.14), we get

$$\begin{aligned} \mathcal{F}_\alpha(t) &= \sum_{k \in \mathbb{Z}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) \mathrm{e}^{\mathrm{i}\frac{u^2}{2}\cot\alpha} \mathrm{e}^{-\mathrm{i}ku} \, \mathrm{d}u \right) \mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha} \mathrm{e}^{\mathrm{i}kt} \\ &= \mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha} \sum_{k \in \mathbb{Z}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(u) \mathrm{e}^{-\mathrm{i}ku} \, \mathrm{d}u \right) \mathrm{e}^{\mathrm{i}kt}, \end{aligned}$$

where

$$g(u) = f(u)\mathrm{e}^{\mathrm{i}\frac{u^2}{2}\cot\alpha} \quad \text{for} \quad u \in [-\pi,\pi].$$

The function $g(u)$ is an intermediary function. Since the derivative of f exists, the derivative of g exists as well. Let $\hat{g}(k)$ be the Fourier coefficients of the function g, then

$$\mathcal{F}_\alpha(t) = \mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\sum_{k\in\mathbb{Z}}\hat{g}(k)\mathrm{e}^{\mathrm{i}kt} = \mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathcal{F}(t),$$

where

$$\mathcal{F}(t) = \sum_{k\in\mathbb{Z}}\hat{g}(k)\mathrm{e}^{\mathrm{i}kt}$$

is the classical Fourier series of g. The uniform convergence of the classical Fourier series of the function g is a standard result in Fourier analysis [41, p. 225]. Now we prove the uniform convergence of FrFS of f, i.e.,

$$\begin{aligned}\lim_{N\to\infty}\max_{-\pi\le t\le\pi}|f(t) - S_{N,\alpha}(t)| &= \max_{-\pi\le t\le\pi}|f(t) - \mathcal{F}_\alpha(t)| \\ &= \max_{-\pi\le t\le\pi}|f(t) - \mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathcal{F}(t)| = 0.\end{aligned}$$

Hence, the partial sum $S_{N,\alpha}(t)$ converges uniformly to $f(t)$. ■

Chapter 3

Estimates of Error Function and Generalized Error Function

In this chapter, we prove some critical inequalities related to error functions of a complex variable and derive the asymptotic expansion of generalized error functions. In Section 3.1, we define the error function of a complex variable and its complement. In Section 3.2, we give an overview of the Landau symbol and define asymptotic series expansion. Further, we give the asymptotic expansion of the error function of a complex variable. We separate the error function into real and imaginary part in Section 3.3. We obtain estimates of the absolute value of the error function of a complex variable in Section 3.4. Moreover, we prove some critical inequalities involving real and imaginary part of the error function. We define generalized error functions in Section 3.5. Further, we simplify the complex exponential integrals which result into generalized error functions. Then we list some properties of generalized error functions. In the last part, we derive the asymptotic approximations of generalized error functions and error estimates of the remainder terms.

3.1 Error Functions on Complex Plane

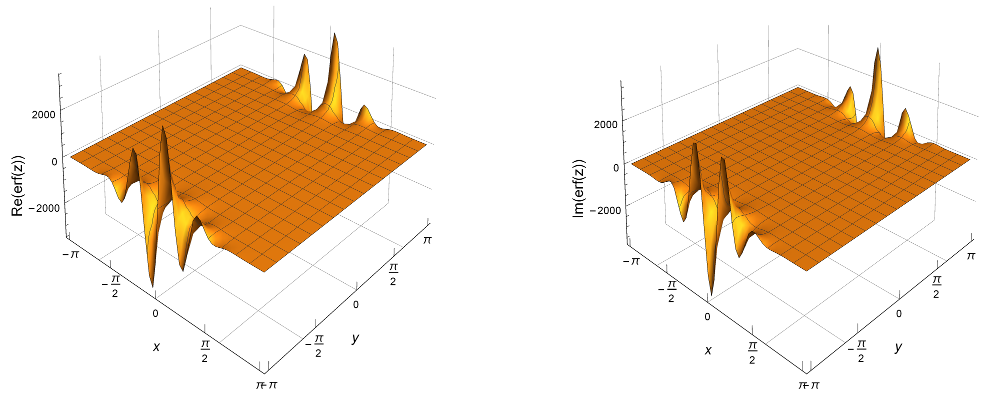
The error functions belong to an important class of special functions of mathematical physics which have integral representations. The error functions of a complex variable play an important role in applied mathematics, physics and engineering problems.

Definition 3.1.1. The error function of a complex variable is defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-u^2}\, \mathrm{d}u, \quad z \in \mathbb{C} \tag{3.1}$$

for an arbitrary integration path in the plane $z = x + \mathrm{i}y$.

The graph of the error function of a complex variable is shown in Figure 3.1.



(a) (b)

Figure 3.1: The error function of a complex variable on $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$; (a) the real part, and (b) the imaginary part.

The error function has the following properties:

i) The error function is zero at origin, i.e., $\operatorname{erf}(0) = 0$.

ii) For all $z \in \mathbb{C}$, the following symmetry relations are satisfied

$$\operatorname{erf}(-z) = -\operatorname{erf}(z) \quad \text{and} \quad \overline{\operatorname{erf}(z)} = \operatorname{erf}(\overline{z}).$$

iii) The limiting value of the error function is given by [48, p. 298]

$$\lim_{z\to\infty} \operatorname{erf}(z) = 1 \quad \text{in the sector} \quad |\arg z| < \frac{\pi}{4}.$$

There is no closed form available for the error function. Since $\operatorname{erf}(z)$ is an entire function, it can be expanded into a convergent power series for all values of z. We expand e^{-u^2} by its power series, and from (3.1) term by term integration yields the power series expansion of the error function

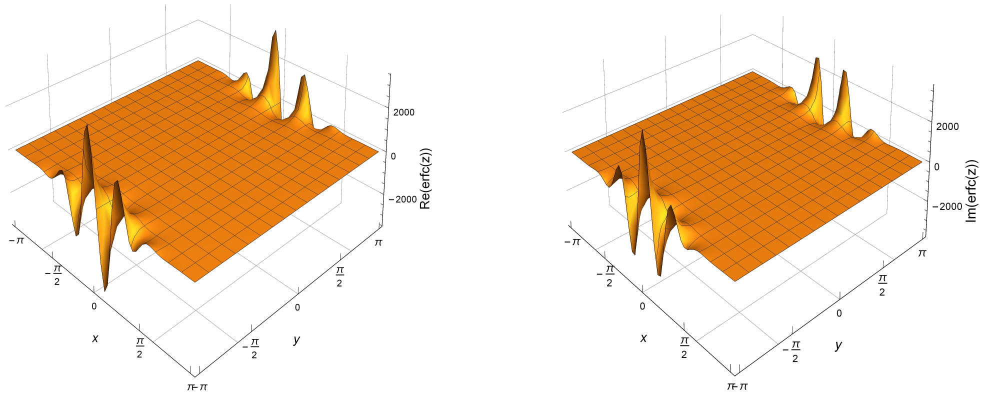
$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!(2k+1)}, \quad |z| < \infty.$$

Definition 3.1.2. The complimentary error function of a complex variable is defined as

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \mathrm{e}^{-u^2}\, \mathrm{d}u, \quad z \in \mathbb{C},$$

where the path of integration is subject to the restriction $\arg u \to \theta$ with $|\theta| < \frac{\pi}{4}$ as $u \to \infty$ along the path [48, p. 297].

The graph of the complementary error function of a complex variable is shown in Figure 3.2.



(a) (b)

Figure 3.2: The complementary error function of a complex variable on $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$; (a) the real part, and (b) the imaginary part.

The complementary error function has the following properties:

i) The complementary error function follows symmetry relation

$$\overline{\operatorname{erfc}(z)} = \operatorname{erfc}(\overline{z}), \quad \text{for all } z \in \mathbb{C}.$$

ii) The relation between the error function and the complementary error function is

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z), \quad \text{for all } z \in \mathbb{C}.$$

Further details on error functions and its complement can be found in the textbooks on special functions of mathematical physics [27, 31, 48, 55, 56].

3.2 Asymptotic Expansion

This section is devoted to an introduction to the asymptotic expansion of the functions of a complex variable. Further we provide the asymptotic expansion for the error function of a complex variable and estimate of the remainder term.

Poincaré introduced the theory of asymptotic expansion in 1886. This asymptotic expansion provides a way to deal with divergent power series the same way as convergent power series. We consider an infinite series (convergent or divergent) which describes a function asymptotically in a neighborhood of a limit point of the function. Thus, any partial sum of the series yields a very good approximation of the function with an error of the order of first neglected terms. The error in approximation is denoted by $\mathcal{O}$ and o symbols introduced by Landau. We follow the definitions and notations given by Erdélyi [7], Copson [25], Bleistein [53] and Wong [64].

Definition 3.2.1. Let S be a sector, $\theta_0 < \arg z < \theta_1$, in the complex plane. Let f and g be two complex valued functions defined on S. Let z_0 be a limit point of S. We write

$$f(z) = \mathcal{O}(g(z)) \quad \text{as } z \to z_0$$

if there is a constant $C > 0$ and a neighborhood U of z_0 such that for all $z \in U \cap S$

$$|f(z)| \leq C\,|g(z)|\,.$$

In other words $\left|\frac{f(z)}{g(z)}\right|$ is bounded or f is of order not more than g.
We also write

$$f(z) = o(g(z)) \quad \text{as } z \to z_0$$

if there is a constant $\delta > 0$ and a neighborhood U of z_0 such that for all $z \in U \cap S$

$$|f(z)| \leq \delta\,|g(z)|\,.$$

In other words $\frac{f(z)}{g(z)}$ approaches zero or f is of order less than g.

To define asymptotic series expansion of a function, we need to describe the asymptotic behavior of a function in terms of an asymptotic sequence.

Definition 3.2.2. Let S be a sector, $\theta_0 < \arg z < \theta_1$, in the complex plane. Let $\{\psi_k : k \geq 0\}$ be a sequence of complex valued functions defined on $S \subset \mathbb{C}$ of the complex plane. Let z_0 be a limit point of S. The sequence of functions $\{\psi_k : k \geq 0\}$ is said to be an asymptotic sequence if for every $k \geq 0$

$$\psi_{k+1}(z) = o\left(\psi_k(z)\right) \quad \text{as } z \to z_0.$$

The series formed by the asymptotic sequence with complex coefficients is an asymptotic series expansion.

Definition 3.2.3. Let $\{\psi_k : k \geq 0\}$ be an asymptotic sequence of functions as $z \to z_0$. The series

$$\sum_{k=0}^{\infty} d_k \psi_k(z)$$

is called asymptotic series expansion of f as $z \to z_0$ if for every value of $N \geq 0$

$$f(z) = \sum_{k=0}^{N} d_k \psi_k(z) + o\left(\psi_N(z)\right) \quad \text{as } z \to z_0. \tag{3.2}$$

The function which possesses asymptotic series expansion is written as

$$f(z) \sim \sum_{k=0}^{\infty} d_k \psi_k(z) \quad \text{as } z \to z_0.$$

We write (3.2) as

$$f(z) = \sum_{k=0}^{N-1} d_k \psi_k(z) + \mathcal{O}\left(\psi_N(z)\right) \quad \text{as } z \to z_0,$$

where the partial sum

$$\sum_{k=0}^{N-1} d_k \psi_k(z)$$

is an approximation to $f(z)$ with an error of order $\mathcal{O}(\psi_N(z))$ as $z \to z_0$. This means the error is of the same order of magnitude as the first term neglected.

Remark. The Poincaré asymptotic expansion of $f(z)$, if exists, is unique. The coefficients are uniquely determined by [53, p. 16]

$$a_N = \lim_{z \to z_0} \frac{1}{\psi_N(z)} \left(f(z) - \sum_{k=0}^{N-1} d_k \psi_k(z) \right).$$

Definition 3.2.4. Let S be a sector, $\theta_0 < \arg z < \theta_1$, in the complex plane. Let f and g be two complex valued functions defined on S with $g \neq 0$ in a neighborhood of z_0 except at z_0. We say $f(z)$ is asymptotically equivalent to $g(z) \neq 0$ if

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 1.$$

Definition 3.2.5. The first non zero term in $d_k \psi_k(z)$ is called dominant term. We write

$$f(z) \sim d_k \psi_k(z) \quad \text{as } z \to z_0.$$

This shows that f behaves like ψ_k at z_0.

3.2.1 Asymptotic Expansion of Error Function

The efficient computation of error functions of a complex variable is an active area of research [34, 43, 73, 76]. There are various methods available for the computation of such functions to high accuracy. The power series expansion of error functions is useful for small values of z. As the values of z are growing, the power series expansion has slow convergence. Therefore, the asymptotic expansions are more practical significant for large values of z. The fast computation of the error function can be obtained by using the asymptotic expansion of the complementary error function in the relation

$$\operatorname{erf}(z) = 1 - \operatorname{erfc}(z).$$

The asymptotic expansion of the complementary error function is given in the following lemma.

Lemma 3.2.1. *The asymptotic expansion of the complementary error function as $z \to \infty$ on the sector $\theta_0 < \arg z < \theta_1$ of the complex plane is given by [54, p. 39]*

$$\operatorname{erfc}(z) = \frac{\mathrm{e}^{-z^2}}{z\sqrt{\pi}} \left(\sum_{l=0}^{n-1} \left(\frac{1}{2}\right)_l \left(\frac{-1}{z^2}\right)^l + \varepsilon_n(z) \right),$$

where the error term $\varepsilon_n(z)$ is bounded [28, p. 111] by

$$|\varepsilon_n(z)| \leq \begin{cases} \left| \left(\frac{1}{2}\right)_n \left(\frac{-1}{z^2}\right)^n \right| & \text{for } |\arg z| \leq \frac{\pi}{2}, \\ \left| \left(\frac{1}{2}\right)_n \left(\frac{-1}{z^2}\right)^n \right| \csc(\arg z) & \text{for } \frac{\pi}{2} \leq \arg z < \pi. \end{cases}$$

The remainder

$$\varepsilon_n(z) = \left(\frac{1}{2}\right)_n \left(\frac{-1}{z^2}\right)^n \int_0^\infty \mathrm{e}^{-t} \left(1 + \frac{t}{z^2}\right)^{-n-\frac{1}{2}} \mathrm{d}t$$

is of order $\mathcal{O}(z^{-n})$ as $z \to \infty$ in $|\arg z| < q\pi$ with $q < 1$.

Remark. In Lemma 3.2.1, the symbol $\left(\frac{1}{2}\right)_n$ is called the Pochhammer symbol. Its value is given by [44, p. 163]

$$\left(\frac{1}{2}\right)_n = \frac{(2n)!}{4^n n!}, \quad n \geq 0.$$

3.3 Real and Imaginary Part of Error Function

Let

$$\operatorname{erf}\left(\pi\sqrt{a - \mathrm{i}b}\right), \quad a > 0,\ b \in \mathbb{R} \tag{3.3}$$

be an error function with the square root $\sqrt{a - \mathrm{i}b}$ evaluated for principal values. We separate the error function of the form (3.3) into real and imaginary part in terms of real line integrals with the oscillating integrands. The real and imaginary part of the error function (3.3) can be evaluated by numerical computation.

Lemma 3.3.1. *Let*

$$\frac{\operatorname{erf}\left(\pi\sqrt{a - \mathrm{i}b}\right)}{\pi\sqrt{a - \mathrm{i}b}}, \quad a,\ b > 0 \tag{3.4}$$

be splitted into real part

$$I_c = \frac{2}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-a\pi^2 u^2} \cos\left(b\pi^2 u^2\right)\ \mathrm{d}u$$

and imaginary part

$$I_s = \frac{2}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-a\pi^2 u^2} \sin\left(b\pi^2 u^2\right) \, \mathrm{d}u$$

respectively. Then both I_c and I_s are positive.

Proof. For a complex variable z, the error function

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-u^2} \, \mathrm{d}u$$

can be considered as a line integral in the complex plane. Since the error function is an entire function of a complex variable, there can be different paths [8, p. 282] chosen to execute this line integral in the complex plane. If we select a linear path from $u = 0$ to $u = z$, the complex error function can be written in the form [44, p. 407]

$$\mathrm{erf}(z) = \frac{2z}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-z^2 u^2} \, \mathrm{d}u.$$

The line integral representation of the error function (3.4) is

$$\frac{\mathrm{erf}\left(\pi\sqrt{a - \mathrm{i}b}\right)}{\pi\sqrt{a - \mathrm{i}b}} = \frac{2\pi\sqrt{a - \mathrm{i}b}}{\pi\sqrt{\pi}\sqrt{a - \mathrm{i}b}} \int_0^1 \mathrm{e}^{-\pi^2(a - \mathrm{i}b)u^2} \, \mathrm{d}u. \tag{3.5}$$

Using Euler's identity

$$\mathrm{e}^{-\pi^2(a-\mathrm{i}b)u^2} = \mathrm{e}^{-a\pi^2u^2} \cos\left(b\pi^2u^2\right) + \mathrm{i}\mathrm{e}^{-a\pi^2u^2} \sin\left(b\pi^2u^2\right),$$

the error function in (3.5) is written as a sum of real and imaginary part

$$\begin{aligned} \frac{\mathrm{erf}\left(\pi\sqrt{a - \mathrm{i}b}\right)}{\pi\sqrt{a - \mathrm{i}b}} &= \frac{2}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-a\pi^2u^2} \cos\left(b\pi^2u^2\right) \, \mathrm{d}u + \mathrm{i}\frac{2}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-a\pi^2u^2} \sin\left(b\pi^2u^2\right) \, \mathrm{d}u \\ &= I_c + \mathrm{i}I_s. \end{aligned}$$

Now we prove that both integrals I_c and I_s are positive.

The integral I_c

Let $I_{c,n}$ be an integral over successive intervals $\left((4n+3)\frac{\pi}{2}, (4n+7)\frac{\pi}{2}\right)$ for all $n \in \mathbb{N}_0$ given by

$$I_{c,n} = \int_{(4n+3)\frac{\pi}{2}}^{(4n+7)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}} \, \mathrm{d}v. \tag{3.6}$$

We prove that $I_{c,n} > 0$ for all $n \in \mathbb{N}_0$. We write the integral (3.6) as

$$\begin{aligned} I_{c,n} &= \int_{(4n+3)\frac{\pi}{2}}^{(4n+5)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v + \int_{(4n+5)\frac{\pi}{2}}^{(4n+7)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v \\ &= \int_{(4n+3)\frac{\pi}{2}}^{(4n+5)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v + \int_{(4n+3)\frac{\pi}{2}}^{(4n+5)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}(v+\pi)} \frac{\cos(v+\pi)}{\sqrt{v+\pi}}\, \mathrm{d}v \\ &= \int_{(4n+3)\frac{\pi}{2}}^{(4n+5)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \left(\frac{\cos v}{\sqrt{v}} - \mathrm{e}^{-\frac{a}{b}\pi} \frac{\cos v}{\sqrt{v+\pi}} \right) \mathrm{d}v. \end{aligned}$$

We know

$$\mathrm{e}^{-\frac{a}{b}v} \left(\frac{\cos v}{\sqrt{v}} - \mathrm{e}^{-\frac{a}{b}\pi} \frac{\cos v}{\sqrt{v+\pi}} \right) > 0$$

for all $v \in \left((4n+3)\frac{\pi}{2}, (4n+5)\frac{\pi}{2}\right)$, and for all $n \in \mathbb{N}_0$. Therefore, $I_{c,n} > 0$ for all $n \in \mathbb{N}_0$. Let I_1 be the integral

$$I_1 = \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v. \tag{3.7}$$

We estimate the integral I_1 in (3.7) as

$$\begin{aligned} I_1 &\geq \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{a\pi}{2b}} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-\frac{a\pi}{2b}} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v \\ &= \mathrm{e}^{-\frac{a\pi}{2b}} \int_0^{\frac{3\pi}{2}} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v = \sqrt{2\pi}\mathrm{e}^{-\frac{a\pi}{2b}} C(\sqrt{3}), \end{aligned}$$

where $C(t)$ is the Fresnel cosine integral defined by

$$C(t) = \int_0^t \cos\left(\frac{\pi}{2}u^2\right) \mathrm{d}u.$$

We expand the Fresnel integral by power series [48, p. 301]

$$I_1 \geq \sqrt{2\pi}\mathrm{e}^{-\frac{a\pi}{2b}} \sum_{l=0}^{\infty} \frac{(-1)^l \left(\frac{\pi}{2}\right)^{2l} \left(\sqrt{3}\right)^{4l+1}}{(2l)!(4l+1)}. \tag{3.8}$$

We prove that for sufficiently large l, the absolute values of the terms of the series (3.8) are monotonic decreasing

$$\begin{aligned} u_{l+1} - u_l &= \frac{\sqrt{3}\left(\frac{\pi}{2}\right)^{2l+2} 3^{2l+2}}{(2l+2)!(4l+5)} - \frac{\sqrt{3}\left(\frac{\pi}{2}\right)^{2l} 3^{2l}}{(2l)!(4l+1)} \\ &= \frac{-\sqrt{3}\left(\frac{\pi}{2}\right)^{2l} 3^{2l}}{(2l+2)!(4l+5)(2l)!(4l+1)} \\ &\times \left(\frac{4l+1}{4}\left((4l+3)^2 - 1 - 9\pi^2\right) + 2(l+1)(2l+1) \right). \end{aligned}$$

Now $u_{l+1} - u_l \le 0$ if

$$\begin{aligned}
&\frac{4l+1}{4}\left((4l+3)^2 - 1 - 9\pi^2\right) + 2(l+1)(2l+1)\\
&= 16l^3 + 32l^2 - 70l - 19 > 0,
\end{aligned}$$

which holds for $l \ge 2$. Therefore, the series (3.8) is an alternating series. Using Leibniz criterion, it is convergent. We split the series (3.8) into a partial sum and the remainder

$$I_1 \ge \sqrt{2\pi}\mathrm{e}^{-\frac{a\pi}{2b}} \left(\sum_{l=0}^{5} \frac{(-1)^l \sqrt{3}\left(\frac{\pi}{2}\right)^{2l} 3^{2l}}{(2l)!(4l+1)} + R_5 \right) \tag{3.9}$$

where

$$R_5 = \sum_{l=5+1}^{\infty} \frac{(-1)^l \sqrt{3}\left(\frac{\pi}{2}\right)^{2l} 3^{2l}}{(2l)!(4l+1)}.$$

The remainder of the alternating series can be bounded by

$$|R_l| \le u_{l+1}.$$

Therefore, the remainder R_5 has bounds

$$|R_5| \le \frac{\sqrt{3}\left(\frac{\pi}{2}\right)^{12} 3^{12}}{12!(25)} < 0.01734 < 0.018.$$

The partial sum in (3.9) is evaluated by using Mathematica 11.3 as

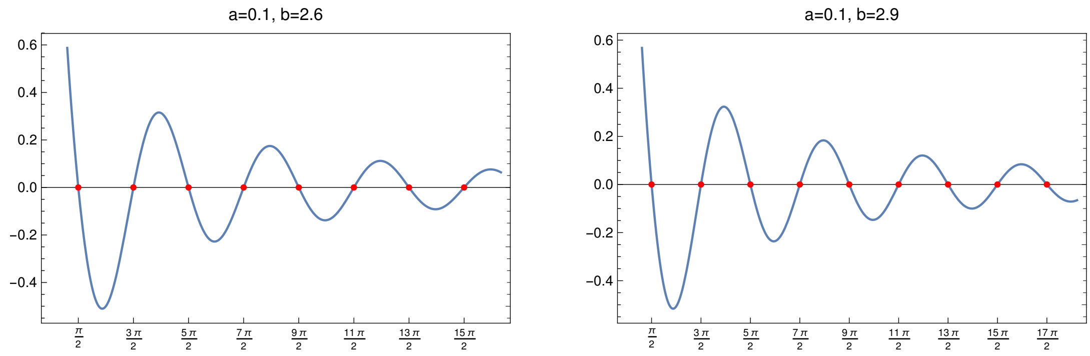
$$\sum_{l=0}^{5} \frac{(-1)^l \sqrt{3}\left(\frac{\pi}{2}\right)^{2l} 3^{2l}}{(2l)!(4l+1)} > 0.3.$$

Hence, $I_1 > 0$. Now we prove for all $a,\ b > 0$, the integral satisfies

$$I_c = \frac{2}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-a\pi^2u^2} \cos\left(b\pi^2u^2\right)\, \mathrm{d}u > 0.$$

Using substitution $v = b\pi^2u^2$, the integral I_c can be written as

$$I_c = \frac{1}{\pi\sqrt{b\pi}} \int_0^{b\pi^2} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}}\, \mathrm{d}v.$$



(a) (b)

Figure 3.3: The graph of the function $\mathrm{e}^{-\frac{a}{b}v}\frac{\cos v}{\sqrt{v}}$ (a) when $a = 0.1,\ b = 2.6$, and (b) when $a = 0.1,\ b = 2.9$.

There are three cases:

i) For $b > \frac{1}{2\pi}$, let $N_c \in \mathbb{Z}^+$ be determined by

$$(4N_c + 1)\frac{\pi}{2} < b\pi^2 \leq (4N_c + 3)\frac{\pi}{2}.$$

We estimate the integral I_c as

$$\begin{aligned}
I_c \geq & \frac{1}{\pi\sqrt{b\pi}} \int_0^{(4N_c+1)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v}\frac{\cos v}{\sqrt{v}}\, \mathrm{d}v \\
= & \frac{1}{\pi\sqrt{b\pi}} \left(\int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v}\frac{\cos v}{\sqrt{v}}\, \mathrm{d}v + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-\frac{a}{b}v}\frac{\cos v}{\sqrt{v}}\, \mathrm{d}v \right. \\
& \left. + \sum_{n=0}^{N_c-2} \int_{(4n+3)\frac{\pi}{2}}^{(4n+7)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v}\frac{\cos v}{\sqrt{v}}\, \mathrm{d}v + \int_{(4N_c-1)\frac{\pi}{2}}^{(4N_c+1)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v}\frac{\cos v}{\sqrt{v}}\, \mathrm{d}v \right) \\
= & \frac{1}{\pi\sqrt{b\pi}} \left(I_1 + \sum_{n=0}^{N_c-2} I_{c,n} + I_{N_c} \right). \qquad (3.10)
\end{aligned}$$

From (3.6), the integrals satisfy $I_{c,n} > 0$ for all $n \in \mathbb{N}_0$. Further $I_1 > 0$ and $I_{N_c} > 0$ for all $N_c \in \mathbb{Z}^+$. Hence, from (3.10)

$$I_c \geq \frac{1}{\pi\sqrt{b\pi}} \left(I_1 + \sum_{n=0}^{N_c-2} I_{c,n} + I_{N_c} \right) > 0.$$

ii) For $b > \frac{1}{2\pi}$, let $N_c \in \mathbb{N}_0$ be determined by

$$(4N_c + 3)\frac{\pi}{2} < b\pi^2 \leq (4N_c + 5)\frac{\pi}{2}.$$

We estimate the integral I_c as

$$\begin{aligned}
I_c \geq & \frac{1}{\pi\sqrt{b\pi}} \int_0^{(4N_c+3)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}} \; \mathrm{d}v \\
= & \frac{1}{\pi\sqrt{b\pi}} \left(\int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}} \; \mathrm{d}v + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}} \; \mathrm{d}v \right. \\
& \left. + \sum_{n=0}^{N_c-1} \int_{(4n+3)\frac{\pi}{2}}^{(4n+7)\frac{\pi}{2}} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}} \; \mathrm{d}v \right) \\
= & \frac{1}{\pi\sqrt{b\pi}} \left(I_1 + \sum_{n=0}^{N_c-1} I_{c,n} \right).
\end{aligned} \tag{3.11}$$

From (3.6), the integrals satisfy $I_{c,n} > 0$ for all $n \in \mathbb{N}_0$. Further the integral $I_1 > 0$. Hence, from (3.11)

$$I_c \geq \frac{1}{\pi\sqrt{b\pi}} \left(I_1 + \sum_{n=0}^{N_c-1} I_{c,n} \right) > 0.$$

iii) For $0 < b \leq \frac{1}{2\pi}$, the integrand satisfies $\mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}} > 0$. Hence

$$I_c = \frac{1}{\pi\sqrt{b\pi}} \int_0^{b\pi^2} \mathrm{e}^{-\frac{a}{b}v} \frac{\cos v}{\sqrt{v}} \; \mathrm{d}v > 0.$$

The integral I_s

Let $I_{s,n}$ be an integral over successive intervals $(2n\pi, (2n+2)\pi)$ for all $n \in \mathbb{N}_0$ given by

$$I_{s,n} = \int_{2n\pi}^{(2n+2)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}} \; \mathrm{d}v. \tag{3.12}$$

We prove that $I_{s,n} > 0$ for all $n \in \mathbb{N}_0$. We write the integral (3.12) into the following form

$$\begin{aligned}
I_{s,n} &= \int_{2n\pi}^{(2n+1)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}} \; \mathrm{d}v + \int_{(2n+1)\pi}^{(2n+2)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}} \; \mathrm{d}v \\
&= \int_{2n\pi}^{(2n+1)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}} \; \mathrm{d}v + \int_{2n\pi}^{(2n+1)\pi} \mathrm{e}^{-\frac{a}{b}(v+\pi)} \frac{\sin(v+\pi)}{\sqrt{v+\pi}} \; \mathrm{d}v \\
&= \int_{2n\pi}^{(2n+1)\pi} \mathrm{e}^{-\frac{a}{b}v} \left(\frac{\sin v}{\sqrt{v}} - \mathrm{e}^{-\frac{a}{b}\pi} \frac{\sin v}{\sqrt{v+\pi}} \right) \; \mathrm{d}v.
\end{aligned}$$

We know

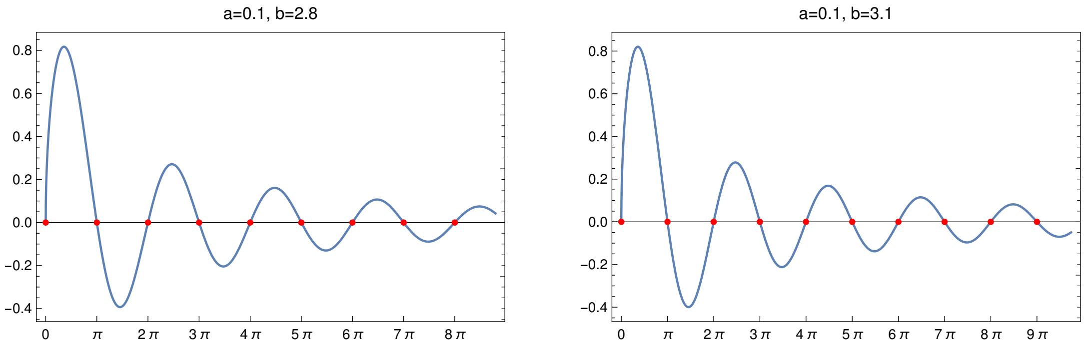
$$\frac{\sin v}{\sqrt{v}} - \mathrm{e}^{-\frac{a}{b}\pi} \frac{\sin v}{\sqrt{v+\pi}} > 0$$

for all $v \in (2n\pi, (2n+1)\pi)$, and for all $n \in \mathbb{N}_0$. Therefore, $I_{s,n} > 0$ for all $n \in \mathbb{N}_0$. Now we prove for all $a, b > 0$, the integral satisfies

$$I_s = \frac{2}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-a\pi^2 u^2} \sin\left(b\pi^2 u^2\right)\, \mathrm{d}u > 0.$$

Using substitution $v = b\pi^2 u^2$, the integral I_s becomes

$$I_s = \frac{1}{\pi\sqrt{b\pi}} \int_0^{b\pi^2} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}}\, \mathrm{d}v.$$



(a) (b)

Figure 3.4: The graph of the function $\mathrm{e}^{-\frac{a}{b}v}\frac{\sin v}{\sqrt{v}}$ (a) when $a = 0.1,\ b = 2.8$, and (b) when $a = 0.1,\ b = 3.1$.

There are three cases:

i) For $b > \frac{1}{\pi}$, let $N_s \in \mathbb{N}_0$ be determined by

$$(2N_s + 1)\pi < b\pi^2 \leq (2N_s + 2)\pi.$$

We estimate the integral I_s as

$$\begin{aligned} I_s &\geq \frac{1}{\pi\sqrt{b\pi}} \int_0^{(2N_s+1)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}}\, \mathrm{d}v \\ &= \frac{1}{\pi\sqrt{b\pi}} \left(\sum_{n=0}^{N_s-1} \int_{2n\pi}^{(2n+2)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}}\, \mathrm{d}v + \int_{2N_s\pi}^{(2N_s+1)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}}\, \mathrm{d}v \right) \\ &= \frac{1}{\pi\sqrt{b\pi}} \left(\sum_{n=0}^{N_s-1} I_{s,n} + I_{N_s} \right). \end{aligned} \tag{3.13}$$

From (3.12), the integrals satisfy $I_{s,n} > 0$ for all $n \in \mathbb{N}_0$ and $I_{N_s} > 0$ for all $N_s \in \mathbb{N}_0$. Hence, from (3.13)

$$I_s \geq \frac{1}{\pi\sqrt{b\pi}} \left(\sum_{n=0}^{N_s-1} I_{s,n} + I_{N_s} \right) > 0.$$

ii) For $b > \frac{1}{\pi}$, let $N_s \in \mathbb{N}_0$ be determined by

$$(2N_s + 2)\pi < b\pi^2 \leq (2N_s + 3)\pi.$$

We estimate the integral I_s as

$$\begin{aligned} I_s &\geq \frac{1}{\pi\sqrt{b\pi}} \int_0^{(2N_s+2)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}}\, \mathrm{d}v \\ &= \frac{1}{\pi\sqrt{b\pi}} \sum_{n=0}^{N_s} \int_{2n\pi}^{(2n+2)\pi} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}}\, \mathrm{d}v = \frac{1}{\pi\sqrt{b\pi}} \sum_{n=0}^{N_s} I_{s,n}. \end{aligned} \tag{3.14}$$

From (3.12), the integrals satisfy $I_{s,n} > 0$ for all $n \in \mathbb{N}_0$. Hence, from (3.14)

$$I_s \geq \frac{1}{\pi\sqrt{b\pi}} \sum_{n=0}^{N_s} I_{s,n} > 0.$$

iii) For $0 < b \leq \frac{1}{\pi}$, the integrand satisfies $\mathrm{e}^{-\frac{a}{b}v}\frac{\sin v}{\sqrt{v}} > 0$. Hence

$$I_s = \frac{1}{\pi\sqrt{b\pi}} \int_0^{b\pi^2} \mathrm{e}^{-\frac{a}{b}v} \frac{\sin v}{\sqrt{v}}\, \mathrm{d}v > 0.$$

■

3.4 Estimates of Error Function Over a Domain

Let Ω_R be a domain defined by

$$\Omega_R = \left\{ (a,b) : a \geq \frac{1}{2},\ 0 < b \leq \frac{\sqrt{a\pi}}{8\pi} \left(\lambda \mathrm{e}^{a\pi^2} + \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi} \right) \right\}, \tag{3.15}$$

where

$$\lambda = \left(1 - \frac{2\sqrt{2}\mathrm{e}^{\frac{-\pi^2}{2}}}{\pi^{\frac{3}{2}}} \right)^2 \approx 0.993. \tag{3.16}$$

In this section, it is convenient to include two results on the domain Ω_R which will be used to prove Theorem 4.3.1 in Chapter 4. The domain Ω_R is illustrated in Figure 3.5.

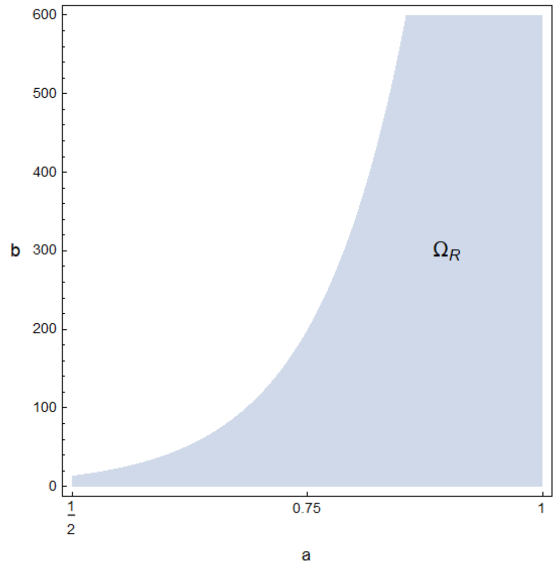


Figure 3.5: The region plot of the domain Ω_R for $\frac{1}{2} \leq a \leq 1$ and $0 \leq b \leq 600$.

3.4.1 Infimum of the Absolute Value of Error Function

We derive an estimate of $\left|\operatorname{erf}\left(\pi\sqrt{a - \mathrm{i}b}\right)\right|^2$. We prove the following lemma.

Lemma 3.4.1. *For the domain* Ω_R *given in* (3.15)*, we have*

$$\inf_{(a,b)\in\Omega_R} \left|\operatorname{erf}\left(\pi\sqrt{a - \mathrm{i}b}\right)\right|^2 \geq \lambda = \left(1 - \frac{2\sqrt{2}\mathrm{e}^{\frac{-\pi^2}{2}}}{\pi^{\frac{3}{2}}}\right)^2.$$

Proof. For any complex number z, we know

$$|\operatorname{erf}(z)| = |1 - \operatorname{erfc}(z)| \geq 1 - |\operatorname{erfc}(z)|,$$

which leads to

$$\inf_{(a,b)\in\Omega_R} |\operatorname{erf}(z)| \geq 1 - \sup_{(a,b)\in\Omega_R} |\operatorname{erfc}(z)|. \tag{3.17}$$

Let $z = x + \mathrm{i}y$, where x and y are real and imaginary part of z respectively. First, we find an estimate of the absolute value of the complementary error function. For a complex number $z = x + \mathrm{i}y$ with $x > 0$, the absolute value of the complementary error function can be estimated [58] as

$$|\operatorname{erfc}(z)| < \frac{\mathrm{e}^{y^2 - x^2}}{x\sqrt{\pi}}\sqrt{1 + \frac{y^2}{x^2}}. \tag{3.18}$$

Let $z = \pi\sqrt{a - \mathrm{i}b}$ with $\mathrm{Re}(z) > 0$. We separate z into real and imaginary part as

$$\pi\sqrt{a-\mathrm{i}b} = \frac{\pi}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a} - \mathrm{i}\frac{\pi}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a} = x + \mathrm{i}y.$$

We simplify

$$y^2 - x^2 = \frac{\pi^2}{2}\left(\sqrt{a^2+b^2}-a\right) - \frac{\pi^2}{2}\left(\sqrt{a^2+b^2}+a\right) = -a\pi^2$$

and

$$\frac{y^2}{x^2} = \frac{\frac{\pi^2}{2}\left(\sqrt{a^2+b^2}-a\right)}{\frac{\pi^2}{2}\left(\sqrt{a^2+b^2}+a\right)} = \frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}+a}.$$

Substituting $y^2 - x^2$ and $\frac{y^2}{x^2}$ into (3.18), we get

$$\begin{aligned}
\left|\mathrm{erfc}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right| &< \frac{\mathrm{e}^{-a\pi^2}}{\left(\frac{\pi}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a}\right)\sqrt{\pi}}\sqrt{1+\frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}+a}} \\
&= \frac{\sqrt{2}\mathrm{e}^{-a\pi^2}}{\pi^{\frac{3}{2}}\left(\sqrt{\sqrt{a^2+b^2}+a}\right)}\sqrt{\frac{\sqrt{a^2+b^2}+a+\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}+a}} \\
&= \frac{2\left(a^2+b^2\right)^{\frac{1}{4}}\mathrm{e}^{-a\pi^2}}{\pi^{\frac{3}{2}}\left(\sqrt{a^2+b^2}+a\right)}.
\end{aligned}$$

Therefore, the inequality (3.17) can be written as

$$\inf_{(a,b)\in\Omega_R}\left|\mathrm{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\right| \geq 1 - \sup_{(a,b)\in\Omega_R} h(a,b), \tag{3.19}$$

where

$$h(a,b) = \frac{2\left(a^2+b^2\right)^{\frac{1}{4}}\mathrm{e}^{-a\pi^2}}{\pi^{\frac{3}{2}}\left(\sqrt{a^2+b^2}+a\right)}.$$

For all $(a,b) > 0$, the derivative of $h(a,b)$ with respect to b is

$$\frac{\partial h(a,b)}{\partial b} = \frac{-b\left(\sqrt{a^2+b^2}-a\right)\mathrm{e}^{-a\pi^2}}{\pi^{\frac{3}{2}}\left(a^2+b^2\right)^{\frac{3}{4}}\left(\sqrt{a^2+b^2}+a\right)} < 0.$$

We write

$$\sup_{(a,b)\in\Omega_R} h(a,b) = \sup_{a\geq\frac{1}{2}} h(a,0).$$

The inequality (3.19) becomes

$$\inf_{(a,b)\in\Omega_R}\left|\mathrm{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\right| \geq 1 - \sup_{a\geq\frac{1}{2}}\left(\frac{2\mathrm{e}^{-a\pi^2}}{\pi\sqrt{\pi a}}\right).$$

Since $h(a,0)$ is a decreasing function of a, its supremum occurs at $a = \frac{1}{2}$. The above inequality becomes

$$\inf_{(a,b)\in\Omega_R} \left|\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\right| \geq 1 - \frac{2\sqrt{2}\mathrm{e}^{\frac{-\pi^2}{2}}}{\pi^{\frac{3}{2}}}.$$

Finally, the infimum of $\left|\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\right|^2$ is given by

$$\inf_{(a,b)\in\Omega_R} \left|\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\right|^2 \geq \left(1 - \frac{2\sqrt{2}\mathrm{e}^{\frac{-\pi^2}{2}}}{\pi^{\frac{3}{2}}}\right)^2 = \lambda.$$

■

3.4.2 Inequalities Involving the Real and Imaginary Part of Error Function

We prove inequalities related to real and imaginary part of the error function.

Lemma 3.4.2. *Let $\xi_1(a,b)$ and $\xi_2(a,b)$ be functions given by*

$$\xi_1(a,b) = \frac{8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}}{b}\left(a\sin b\pi^2 + b\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,\mathrm{d}u$$

and

$$\xi_2(a,b) = \frac{8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}}{b}\left(b\sin b\pi^2 - a\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u.$$

For $(a,b)\in\Omega_R$ given in (3.15), the following inequalities hold:

$$\sup_{(a,b)\in\Omega_R} \xi_1(a,b) < \lambda \tag{3.20}$$

and

$$\sup_{(a,b)\in\Omega_R} \xi_2(a,b) < \lambda, \tag{3.21}$$

where λ is given in (3.16).

Proof. We divide the domain Ω_R into two parts

$$\Omega_R = \Omega_1 \cup \Omega_2$$

with

$$\Omega_1 = \left\{(a,b) : a \geq \frac{1}{2},\ 0 < b \leq \sqrt{\frac{a\left(\lambda\mathrm{e}^{a\pi^2}\right)^2}{16\pi(a\pi^2+1)^2} - a^2}\right\}$$

and

$$\Omega_2 = \left\{ (a,b) : a \geq \frac{1}{2},\ \sqrt{\frac{a\left(\lambda \mathrm{e}^{a\pi^2}\right)^2}{16\pi(a\pi^2+1)^2} - a^2} < b \leq \Psi(a) \right\},$$

where

$$\Psi(a) = \frac{\sqrt{a\pi}}{8\pi}\left(\lambda \mathrm{e}^{a\pi^2} + \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right),$$

and λ is given in (3.16).

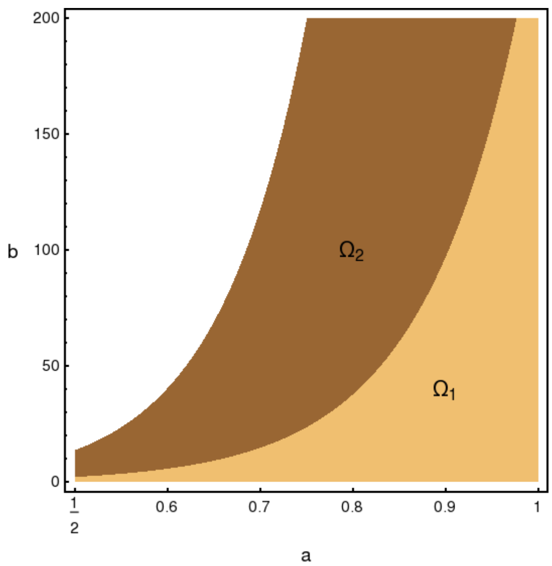


Figure 3.6: The region plot of the domain Ω_R for $\frac{1}{2} \leq a \leq 1$ and $0 \leq b \leq 200$ indicating the domains Ω_1 and Ω_2.

The domain Ω_1

First, we prove Lemma 3.4.2 for domain Ω_1. We write the inequalities (3.20) and (3.21) into the following form

$$\sup_{(a,b)\in\Omega_1}\left(8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}\left(a\pi^2 \operatorname{sinc} b\pi^2 + \cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\ \mathrm{d}u\right) < \lambda \tag{3.22}$$

and

$$\sup_{(a,b)\in\Omega_1}\left(8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}\left(\sin b\pi^2 - \frac{a}{b}\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\ \mathrm{d}u\right) < \lambda, \tag{3.23}$$

where $\operatorname{sinc}(t)$ is defined as

$$\operatorname{sinc}(t) = \begin{cases} \dfrac{\sin \pi t}{\pi t}, & \text{if } t \neq 0, \\ 1, & \text{if } t = 0. \end{cases}$$

There are two cases:

i) Let $a\pi^2 \operatorname{sinc} b\pi^2 + \cos b\pi^2 \leq 0$. From Lemma 3.3.1, the integral satisfies

$$\int_0^1 \mathrm{e}^{-a\pi^2 u^2} \cos\left(b\pi^2 u^2\right) \, \mathrm{d}u > 0.$$

Therefore, the inequality (3.22) holds.

ii) Let $a\pi^2 \operatorname{sinc} b\pi^2 + \cos b\pi^2 > 0$. For all $(a, b) \in \Omega_1$, the following inequality holds

$$a^2 + b^2 < \frac{a\left(\lambda \mathrm{e}^{a\pi^2}\right)^2}{16\pi(a\pi^2 + 1)^2}.$$

Taking square root on both sides, we get

$$\sqrt{a^2 + b^2} < \frac{\sqrt{a}\lambda \mathrm{e}^{a\pi^2}}{4\sqrt{\pi}(a\pi^2 + 1)},$$

which is equivalent to

$$a\pi^2 + 1 < \frac{\sqrt{a}\lambda \mathrm{e}^{a\pi^2}}{4\sqrt{\pi}\sqrt{a^2 + b^2}}.$$

Since $a\pi^2 \operatorname{sinc} b\pi^2 + \cos b\pi^2 < a\pi^2 + 1$, the above inequality becomes

$$a\pi^2 \operatorname{sinc} b\pi^2 + \cos b\pi^2 < \frac{\sqrt{a}\lambda \mathrm{e}^{a\pi^2}}{4\sqrt{\pi}\sqrt{a^2 + b^2}}.$$

Multiplying both sides by $\frac{1}{2\sqrt{a\pi}}$, we get

$$\frac{1}{2\sqrt{a\pi}}\left(a\pi^2 \operatorname{sinc} b\pi^2 + \cos b\pi^2\right) < \frac{\lambda \mathrm{e}^{a\pi^2}}{8\pi\sqrt{a^2 + b^2}}. \tag{3.24}$$

Now

$$\frac{1}{2\sqrt{a\pi}} = \int_0^\infty \mathrm{e}^{-a\pi^2 u^2} \, \mathrm{d}u > \int_0^1 \mathrm{e}^{-a\pi^2 u^2} \, \mathrm{d}u > \int_0^1 \mathrm{e}^{-a\pi^2 u^2} \cos\left(b\pi^2 u^2\right) \, \mathrm{d}u.$$

Then the inequality (3.24) becomes

$$\left(a\pi^2 \operatorname{sinc} b\pi^2 + \cos b\pi^2\right) \int_0^1 \mathrm{e}^{-\pi^2 a u^2} \cos\left(b\pi^2 u^2\right) \, \mathrm{d}u < \frac{\lambda \mathrm{e}^{a\pi^2}}{8\pi\sqrt{a^2 + b^2}},$$

which is equivalent to

$$8\pi\sqrt{a^2 + b^2}\mathrm{e}^{-a\pi^2}\left(a\pi^2 \operatorname{sinc} b\pi^2 + \cos b\pi^2\right) \int_0^1 \mathrm{e}^{-a\pi^2 u^2} \cos\left(b\pi^2 u^2\right) \, \mathrm{d}u < \lambda.$$

Now we prove the inequality (3.23). There are two cases:

i) Let $\sin b\pi^2 - \frac{a}{b}\cos b\pi^2 \leq 0$. From Lemma 3.3.1, the integral satisfies

$$\int_0^1 \mathrm{e}^{-a\pi^2 u^2} \sin\left(b\pi^2 u^2\right)\, \mathrm{d}u > 0.$$

Therefore, the inequality (3.23) holds.

ii) Let $\sin b\pi^2 - \frac{a}{b}\cos b\pi^2 > 0$. There are two cases:

- when $b > \frac{1}{\pi^2}$. For all $(a,b) \in \Omega_1$, the following inequality holds

$$a^2 + b^2 < \frac{a\left(\lambda \mathrm{e}^{a\pi^2}\right)^2}{16\pi(a\pi^2+1)^2} < \frac{b^2 a\left(\lambda \mathrm{e}^{a\pi^2}\right)^2}{16\pi(b+a)^2}. \tag{3.25}$$

  From the inequality (3.25), we write

$$a^2 + b^2 < \frac{b^2 a\left(\lambda \mathrm{e}^{a\pi^2}\right)^2}{16\pi(b+a)^2}$$

  Since $b > 0$, we write the above inequality into following form

$$a^2 + b^2 < \frac{a\left(\lambda \mathrm{e}^{a\pi^2}\right)^2}{16\pi\left(1+\frac{a}{b}\right)^2}.$$

  Taking square root on both sides, we have

$$\sqrt{a^2+b^2} < \frac{\sqrt{a}\lambda \mathrm{e}^{a\pi^2}}{4\sqrt{\pi}\left(1+\frac{a}{b}\right)},$$

  which leads to the following inequality

$$\left(1+\frac{a}{b}\right) < \frac{\sqrt{a}\lambda \mathrm{e}^{a\pi^2}}{4\sqrt{\pi}\sqrt{a^2+b^2}}.$$

  Since $\sin b\pi^2 - \frac{a}{b}\cos b\pi^2 < 1 + \frac{a}{b}$, the above inequality becomes

$$\sin b\pi^2 - \frac{a}{b}\cos b\pi^2 < \frac{\sqrt{a}\lambda \mathrm{e}^{a\pi^2}}{4\sqrt{\pi}\sqrt{a^2+b^2}}.$$

  Multiplying both sides by $\frac{1}{2\sqrt{a\pi}}$, we get

$$\frac{1}{2\sqrt{a\pi}}\left(\sin b\pi^2 - \frac{a}{b}\cos b\pi^2\right) < \frac{\lambda \mathrm{e}^{a\pi^2}}{8\pi\sqrt{a^2+b^2}}. \tag{3.26}$$

Now

$$\frac{1}{2\sqrt{a\pi}} = \int_0^\infty \mathrm{e}^{-a\pi^2u^2}\,\mathrm{d}u > \int_0^1 \mathrm{e}^{-a\pi^2u^2}\,\mathrm{d}u > \int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u.$$

Then the inequality (3.26) becomes

$$\left(\sin b\pi^2 - \frac{a}{b}\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u < \frac{\lambda\mathrm{e}^{a\pi^2}}{8\pi\sqrt{a^2+b^2}},$$

which is equivalent to

$$8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}\left(\sin b\pi^2 - \frac{a}{b}\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u < \lambda.$$

- When $0 < b \leq \frac{1}{\pi^2}$, then we show

$$\sin b\pi^2 - \frac{a}{b}\cos b\pi^2 < 0.$$

The expression is equivalently written as

$$\sin b\pi^2 < \frac{a}{b}\cos b\pi^2.$$

Since $\sin b\pi^2 < b\pi^2$, we have

$$\cos b\pi^2 > \frac{b^2\pi^2}{a}.$$

For $0 < b \leq \frac{1}{\pi^2}$ and $a = \frac{1}{2}$, we see that

$$\cos 1 > 0.53 > 0.21 > \frac{2}{\pi^2}.$$

Therefore, the inequality (3.23) is true.

The domain Ω_2

Using the trigonometric identities

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\cos(\theta-\beta), \quad \beta = \arctan\left(\frac{a}{b}\right)$$

and

$$b\sin\theta - a\cos\theta = \sqrt{a^2+b^2}\sin(\theta-\beta), \quad \beta = \arctan\left(\frac{a}{b}\right),$$

we write the inequalities (3.20) and (3.21) into the following form

$$\sup_{(a,b)\in\Omega_2}\left(\frac{8\pi(a^2+b^2)\mathrm{e}^{-a\pi^2}}{b}\cos\left(b\pi^2-\beta\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,\mathrm{d}u\right) < \lambda \tag{3.27}$$

and

$$\sup_{(a,b)\in\Omega_2}\left(\frac{8\pi(a^2+b^2)\mathrm{e}^{-a\pi^2}}{b}\sin\left(b\pi^2-\beta\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u\right)<\lambda. \tag{3.28}$$

First, we prove the inequality (3.27). There are two cases:

i) Let $\cos\left(b\pi^2-\beta\right)\le 0$. From Lemma 3.3.1, the integral satisfies

$$\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,\mathrm{d}u>0.$$

Therefore, the inequality (3.27) holds.

ii) Let $\cos\left(b\pi^2-\beta\right)>0$. Let Ω_G be a domain given by

$$\Omega_G=\left\{(a,b):a\ge\frac{1}{2},\ \Phi(a)<b<\Psi(a)\right\} \tag{3.29}$$

with

$$\Phi(a)=\frac{\sqrt{a\pi}}{8\pi}\left(\lambda\mathrm{e}^{a\pi^2}-\sqrt{\left(\lambda\mathrm{e}^{a\pi^2}\right)^2-64a\pi}\right)$$

and

$$\Psi(a)=\frac{\sqrt{a\pi}}{8\pi}\left(\lambda\mathrm{e}^{a\pi^2}+\sqrt{\left(\lambda\mathrm{e}^{a\pi^2}\right)^2-64a\pi}\right).$$

We prove the inequality (3.27) for the domain Ω_G given in (3.29). For all $(a,b)\in\Omega_G$, the following inequality holds

$$\frac{\sqrt{a\pi}}{8\pi}\left(\lambda\mathrm{e}^{a\pi^2}-\sqrt{\left(\lambda\mathrm{e}^{a\pi^2}\right)^2-64a\pi}\right)<b<\frac{\sqrt{a\pi}}{8\pi}\left(\lambda\mathrm{e}^{a\pi^2}+\sqrt{\left(\lambda\mathrm{e}^{a\pi^2}\right)^2-64a\pi}\right)$$

which is a solution to the following inequality

$$a^2+b^2<\frac{\lambda\mathrm{e}^{a\pi^2}\sqrt{a\pi}b}{4\pi}.$$

Dividing both sides by $\sqrt{a^2+b^2}$, we get

$$\sqrt{a^2+b^2}<\frac{\lambda\mathrm{e}^{a\pi^2}\sqrt{a\pi}b}{4\pi\sqrt{a^2+b^2}}. \tag{3.30}$$

Since $\sqrt{a^2+b^2}\cos\left(b\pi^2-\beta\right)\le\sqrt{a^2+b^2}$, the inequality (3.30) becomes

$$\sqrt{a^2+b^2}\cos\left(b\pi^2-\beta\right)<\frac{\lambda\mathrm{e}^{a\pi^2}\sqrt{a\pi}b}{4\pi\sqrt{a^2+b^2}}.$$

Multiplying both sides by $\frac{1}{2\sqrt{a\pi}}$, we get

$$\frac{1}{2\sqrt{a\pi}}\sqrt{a^2+b^2}\cos\left(b\pi^2-\beta\right) < \frac{\lambda e^{a\pi^2}b}{8\pi\sqrt{a^2+b^2}}. \tag{3.31}$$

Now

$$\frac{1}{2\sqrt{a\pi}} = \int_0^\infty e^{-a\pi^2u^2}\,du > \int_0^1 e^{-a\pi^2u^2}\,du > \int_0^1 e^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,du.$$

Then the inequality (3.31) becomes

$$\left(a\sin b\pi^2 + b\cos b\pi^2\right)\int_0^1 e^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,du < \frac{\lambda e^{a\pi^2}b}{8\pi\sqrt{a^2+b^2}},$$

which is equivalent to

$$\frac{8\pi\sqrt{a^2+b^2}e^{-a\pi^2}}{b}\left(a\sin b\pi^2 + b\cos b\pi^2\right)\int_0^1 e^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,du < \lambda.$$

Now we prove the inequality (3.28). There are two cases:

i) Let $\cos(b\pi^2-\beta)\leq 0$. From Lemma 3.3.1, the integral satisfies

$$\int_0^1 e^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,du > 0.$$

Therefore, the inequality (3.28) holds.

ii) Let $\sin(b\pi^2-\beta) > 0$. We prove the inequality (3.28). For $(a,b)\in\Omega_G$, the inequality (3.30) is satisfied

$$\sqrt{a^2+b^2} < \frac{\lambda e^{a\pi^2}\sqrt{a\pi}b}{4\pi\sqrt{a^2+b^2}}.$$

Since $\sqrt{a^2+b^2}\sin(b\pi^2-\beta)\leq\sqrt{a^2+b^2}$, the above inequality becomes

$$\sqrt{a^2+b^2}\sin\left(b\pi^2-\beta\right) < \frac{\lambda e^{a\pi^2}\sqrt{a\pi}b}{4\pi\sqrt{a^2+b^2}}.$$

Multiplying both sides by $\frac{1}{2\sqrt{a\pi}}$, we get

$$\frac{1}{2\sqrt{a\pi}}\sqrt{a^2+b^2}\sin\left(b\pi^2-\beta\right) < \frac{\lambda e^{a\pi^2}b}{8\pi\sqrt{a^2+b^2}}. \tag{3.32}$$

Now

$$\frac{1}{2\sqrt{a\pi}} = \int_0^\infty e^{-a\pi^2u^2}\,du > \int_0^1 e^{-a\pi^2u^2}\,du > \int_0^1 e^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,du.$$

Then inequality (3.32) becomes

$$\left(b\sin b\pi^2 - a\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u < \frac{\lambda \mathrm{e}^{a\pi^2}b}{8\pi\sqrt{a^2+b^2}},$$

which is equivalent to

$$\frac{8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}}{b}\left(b\sin b\pi^2 - a\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u < \lambda.$$

Hence, we have proved the inequalities (3.27) and (3.28) for the domain Ω_G. To finish the proof, we show that $\Omega_2 \subset \Omega_G$. We need to prove that

$$\sqrt{\frac{a\left(\lambda \mathrm{e}^{a\pi^2}\right)^2}{16\pi(a\pi^2+1)^2} - a^2} > \frac{\sqrt{a\pi}}{8\pi}\left(\lambda \mathrm{e}^{a\pi^2} - \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right).$$

Let $f_G(a)$ be a function given by

$$f_G(a) = \sqrt{\frac{a\left(\lambda \mathrm{e}^{a\pi^2}\right)^2}{16\pi(a\pi^2+1)^2} - a^2} - \frac{\sqrt{a\pi}}{8\pi}\left(\lambda \mathrm{e}^{a\pi^2} - \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right).$$

We shall show that $f_G(a) > 0$. We write $f_G(a)$ into the following form

$$f_G(a) = \frac{\sqrt{a}}{4\sqrt{\pi}}\left(\frac{\sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 16a\pi(a\pi^2+1)^2}}{(a\pi^2+1)} - \frac{1}{2}\left(\lambda \mathrm{e}^{a\pi^2} - \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right)\right).$$

Writing the right side of the function as a single fraction, we have

$$\begin{aligned} f_G(a) =& \frac{\sqrt{a}}{8\sqrt{\pi}(a\pi^2+1)} \\ &\times\left(2\sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 16a\pi(a\pi^2+1)^2} - (a\pi^2+1)\left(\lambda \mathrm{e}^{a\pi^2} - \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right)\right). \end{aligned}$$

Now $f_G(a) > 0$ iff

$$2\sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 16a\pi(a\pi^2+1)^2} - (a\pi^2+1)\left(\lambda \mathrm{e}^{a\pi^2} - \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right) > 0.$$

We write equivalently as

$$2\sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 16a\pi(a\pi^2+1)^2} > (a\pi^2+1)\left(\lambda \mathrm{e}^{a\pi^2} - \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right).$$

It is sufficient to prove

$$4\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi(a\pi^2+1)^2 > (a\pi^2+1)^2\left(\lambda \mathrm{e}^{a\pi^2} - \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right)^2.$$

Expanding the square on right side, we have the following inequality

$$4\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi(a\pi^2+1)^2$$
$$> (a\pi^2+1)^2\left(2\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi - 2\lambda \mathrm{e}^{a\pi^2}\sqrt{(\lambda \mathrm{e}^{a\pi^2})^2 - 64a\pi}\right).$$

Re-arranging the terms and simplification gives

$$2\lambda \mathrm{e}^{a\pi^2}(a\pi^2+1)^2\sqrt{\lambda^2\mathrm{e}^{2a\pi^2} - 64a\pi} > 2\lambda^2\mathrm{e}^{2a\pi^2}\left((a\pi^2+1)^2 - 2\right).$$

Taking square on both sides, we get

$$4\mathrm{e}^{2a\pi^2}(a\pi^2+1)^4\left(\lambda^2\mathrm{e}^{2a\pi^2} - 64a\pi\right) > 4\lambda^2\mathrm{e}^{4a\pi^2}\left((a\pi^2+1)^2 - 2\right)^2.$$

Expanding the square on right side and simplification of terms gives

$$16\lambda^2\mathrm{e}^{4a\pi^2}(a\pi^2+1)^2 > 16\lambda^2\mathrm{e}^{4a\pi^2} + 256\mathrm{e}^{2a\pi^2}a\pi(a\pi^2+1)^4.$$

Dividing both sides by $16\mathrm{e}^{2a\pi^2}$, we have

$$\lambda^2\mathrm{e}^{2a\pi^2}(a\pi^2+1)^2 > \lambda^2\mathrm{e}^{2a\pi^2} + 16a\pi(a\pi^2+1)^4.$$

Using $(a\pi^2+1)^2 = a^2\pi^4 + 1 + 2a\pi^2$, the above inequality reduced to

$$\lambda^2\mathrm{e}^{2a\pi^2}(a^2\pi^4 + 2a\pi^2) > 16a\pi(a\pi^2+1)^4.$$

Dividing both sides by $a\pi$, we get

$$\lambda^2\mathrm{e}^{2a\pi^2}(a\pi^3 + 2\pi) > 16(a\pi^2+1)^4,$$

which is equivalent to

$$\mathrm{e}^{2a\pi^2} > \frac{16}{\lambda^2\pi}(a\pi^2+1)^3.$$

It remains to prove the above inequality. Let $g(a) = \mathrm{e}^{2a\pi^2}$ and $h(a) = \frac{16}{\lambda^2\pi}(a\pi^2+1)^3$. Then for $a \geq \frac{1}{2}$, the k-th derivative of both $g(a)$ and $h(a)$ are given by

$$g^{(k)}(a) = \left(2\pi^2\right)^k \mathrm{e}^{2a\pi^2}, \quad k \in \mathbb{N}_0,$$

and

$$h^{(k)}(a) = \begin{cases} \dfrac{16\pi^{2k-1}(3!)}{\lambda^2(3-k)!}(a\pi^2+1)^{3-k}, & \text{for } k \leq 3, \\ 0, & \text{for } k > 3. \end{cases}$$

Since it holds numerically

$$\begin{aligned} g^{(0)}\left(\frac{1}{2}\right) &> \quad 19333 > \ 1081 > \ h^{(0)}\left(\frac{1}{2}\right) \\ g^{(1)}\left(\frac{1}{2}\right) &> \ 381632 > \ 5390 > \ h^{(1)}\left(\frac{1}{2}\right) \\ g^{(2)}\left(\frac{1}{2}\right) &> 7.5 \times 10^6 > 17928 > \ h^{(2)}\left(\frac{1}{2}\right) \\ g^{(3)}\left(\frac{1}{2}\right) &> 1.4 \times 10^8 > 29812 > h^{(3)}\left(\frac{1}{2}\right), \end{aligned}$$

by Taylor series expansion, we have

$$g(a) = \sum_{k=0}^{\infty} \frac{g^{(k)}\left(\frac{1}{2}\right)}{k!}\left(a - \frac{1}{2}\right)^k > \sum_{k=0}^{3} \frac{h^{(k)}\left(\frac{1}{2}\right)}{k!}\left(a - \frac{1}{2}\right)^k = h(a).$$

Therefore, the function $f_G(a) > 0$ for all $a \geq \frac{1}{2}$. Hence, $\Omega_2 \subset \Omega_G$. ■

3.5 Generalized Error Functions

In this section, we define a generalized error function and represent complex exponential integral in terms of the generalized error function. Further, we list some properties of generalized error functions which include symmetric properties and complex conjugate properties. Finally, we derive an expression for the derivative of the generalized error function.

The term generalized error function is often used for the n-th power of the error function [29, 33, 47]. We follow the definition and notation of generalized error functions used by Wolfram Mathematica on the mathematical functions site [81].

Definition 3.5.1. The generalized error function is defined as a difference of two error functions

$$\operatorname{erf}(z_1, z_2) = \operatorname{erf}(z_2) - \operatorname{erf}(z_1) \quad \text{for all } z_1, z_2 \in \mathbb{C}. \tag{3.33}$$

Remark. Using the property $\operatorname{erf}(z) = 1 - \operatorname{erfc}(z)$ to (3.33), the generalized error function can be written as a difference of two complementary error functions

$$\operatorname{erf}(z_1, z_2) = \operatorname{erfc}(z_1) - \operatorname{erfc}(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

3.5.1 Complex Exponential Integrals

We consider the integrals of the form

$$\int_{-\pi}^{\pi} \mathrm{e}^{-at^2}\mathrm{e}^{-\mathrm{i}bt^2}\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t, \quad a > 0,\ b \in \mathbb{R},\ k \in \mathbb{Z}. \tag{3.34}$$

The fractional Fourier coefficients of exponential linear chirps are integrals (3.34) which are highly oscillatory. The usefulness of FrFS depends upon the computation of such integrals. These integrals can not have any closed form; the most useful way is to express in terms of a generalized error function of a complex variable.

Lemma 3.5.1. *The integral I, given by*

$$I = \int_{-\pi}^{\pi} \mathrm{e}^{-at^2}\mathrm{e}^{-\mathrm{i}bt^2}\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t, \quad a > 0,\ b \in \mathbb{R},\ k \in \mathbb{Z}, \tag{3.35}$$

can be expressed in terms of a generalized error function of the form

$$I = \frac{\sqrt{\pi}\mathrm{e}^{\frac{-k^2}{4(a+\mathrm{i}b)}}}{2\sqrt{a+\mathrm{i}b}}\,\mathrm{erf}\left(\frac{\mathrm{i}\,(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}\,(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right).$$

Proof. First, we simplify the exponents of exponentials in integral (3.35) by completing square as

$$\begin{aligned}-at^2 - \mathrm{i}bt^2 - \mathrm{i}kt &= -(a+\mathrm{i}b)\left[\left(t + \frac{\mathrm{i}k}{2(a+\mathrm{i}b)}\right)^2 + \frac{k^2}{4(a+\mathrm{i}b)^2}\right] \\ &= \frac{-\left(2(a+\mathrm{i}b)t+\mathrm{i}k\right)^2}{4(a+\mathrm{i}b)} - \frac{k^2}{4(a+\mathrm{i}b)}.\end{aligned}$$

The integral (3.35) becomes

$$I = \mathrm{e}^{\frac{-k^2}{4(a+\mathrm{i}b)}} \int_{-\pi}^{\pi} \mathrm{e}^{\frac{-(2(a+\mathrm{i}b)t+\mathrm{i}k)^2}{4(a+\mathrm{i}b)}}\,\mathrm{d}t. \tag{3.36}$$

The integral (3.36) is evaluated by the substitution method. Let

$$u = 2(a+\mathrm{i}b)t + \mathrm{i}k,$$

which gives

$$\mathrm{d}t = \frac{\mathrm{d}u}{2\,(a+\mathrm{i}b)}.$$

Now the new limits are $u_1 = \mathrm{i}\,(k-2\pi b) - 2\pi a$ and $u_2 = \mathrm{i}\,(k+2\pi b) + 2\pi a$. Therefore, the integral (3.36) becomes

$$I = \frac{\mathrm{e}^{\frac{-k^2}{4(a+\mathrm{i}b)}}}{2\,(a+\mathrm{i}b)} \int_{u_1}^{u_2} \mathrm{e}^{-\frac{1}{4(a+\mathrm{i}b)}u^2}\mathrm{d}u. \tag{3.37}$$

Using the formula [48, p. 303]

$$\int_p^q \mathrm{e}^{-ct^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2\sqrt{c}}\left(\mathrm{erf}\left(q\sqrt{c}\right) - \mathrm{erf}\left(p\sqrt{c}\right)\right),$$

where erf(z) is the error function defined on the whole complex plane, the integral (3.37) is written as

$$I = \frac{\sqrt{\pi}}{2} \frac{\mathrm{e}^{\frac{-k^2}{4(a+\mathrm{i}b)}}}{\sqrt{a+\mathrm{i}b}} \left[\operatorname{erf}\left(\frac{u_2}{2\sqrt{a+\mathrm{i}b}} \right) - \operatorname{erf}\left(\frac{u_1}{2\sqrt{a+\mathrm{i}b}} \right) \right]. \tag{3.38}$$

Replacing the values of u_1 and u_2, (3.38) becomes

$$I = \frac{\sqrt{\pi}}{2} \frac{\mathrm{e}^{\frac{-k^2}{4(a+\mathrm{i}b)}}}{\sqrt{a+\mathrm{i}b}} \left[\operatorname{erf}\left(\frac{\mathrm{i}\,(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) - \operatorname{erf}\left(\frac{\mathrm{i}\,(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \right]. \tag{3.39}$$

We write (3.39) in terms of a generalized error function

$$I = \frac{\sqrt{\pi}\mathrm{e}^{\frac{-k^2}{4(a+\mathrm{i}b)}}}{2\sqrt{a+\mathrm{i}b}} \operatorname{erf}\left(\frac{\mathrm{i}\,(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}\,(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}} \right). \tag{3.40}$$

■

3.5.2 Properties of Generalized Error Function

We list two useful properties of the generalized error function given in (3.40).

Lemma 3.5.2. ***(Conjugate Property)*** *For all $a > 0,\ b \in \mathbb{R}$ and $k \in \mathbb{Z}$, the generalized error function*

$$\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \tag{3.41}$$

is a complex conjugate of the generalized error function

$$\operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)-2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b)+2\pi a}{2\sqrt{a-\mathrm{i}b}} \right).$$

Proof. Since $\operatorname{erf}(z_1, z_2) = \operatorname{erf}(z_2) - \operatorname{erf}(z_1)$, we write the generalized error function (3.41) as a difference of two error functions

$$\begin{aligned} &\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \\ &= \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) - \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}} \right). \end{aligned} \tag{3.42}$$

Taking complex conjugate on both sides in (3.42) gives

$$\begin{aligned} &\overline{\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}} \right)} \\ &= \overline{\operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}} \right)} - \overline{\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}} \right)}. \end{aligned}$$

Using the symmetry properties $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ and $\overline{\operatorname{erf}(z)} = \operatorname{erf}(\overline{z})$ of the error function of a complex variable, it follows

$$\begin{aligned}
&\overline{\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)} \\
&= \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)-2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b)+2\pi a}{2\sqrt{a-\mathrm{i}b}}\right).
\end{aligned}$$

■

Lemma 3.5.3. ***(Symmetry Property)*** *For all $a > 0,\ b \in \mathbb{R}$ and $k \in \mathbb{Z}$, the generalized error function*

$$\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) \tag{3.43}$$

and its complex conjugate

$$\operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)-2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b)+2\pi a}{2\sqrt{a-\mathrm{i}b}}\right) \tag{3.44}$$

are both even functions of k.

Proof. We write the generalized error function (3.43) as a difference of two error functions

$$\begin{aligned}
&\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) \\
&= \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) - \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}\right).
\end{aligned} \tag{3.45}$$

Replacing k by $-k$ in (3.45), we write

$$\begin{aligned}
&\operatorname{erf}\left(\frac{\mathrm{i}(-k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(-k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) \\
&= \operatorname{erf}\left(\frac{\mathrm{i}(-k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) - \operatorname{erf}\left(\frac{\mathrm{i}(-k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}\right).
\end{aligned}$$

We use symmetry properties $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ and $\overline{\operatorname{erf}(z)} = \operatorname{erf}(\overline{z})$ of the error function of a complex variable

$$\begin{aligned}
&\operatorname{erf}\left(\frac{\mathrm{i}(-k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(-k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) \\
&= \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) - \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) \\
&= \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right).
\end{aligned}$$

Therefore, the generalized error function (3.43) is an even function of k. Similarly, we can prove that the generalized error function (3.44) is an even function of k. ■

We derive an expression for the derivative of the generalized error function.

3.5.3 Derivative of Generalized Error Function

Lemma 3.5.4. *The derivative of the generalized error function with respect to variable b is given by*

$$\frac{\mathrm{d}}{\mathrm{d}b}\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}},\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)$$
$$=\frac{\sqrt{a+\mathrm{i}b}(a-\mathrm{i}b)^2}{2\sqrt{\pi}(a^2+b^2)^2}\left(((k-2\pi b)+\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{((k+2\pi b)-\mathrm{i}2\pi a)^2}{4(a+\mathrm{i}b)}}-((k+2\pi b)-\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{((k-2\pi b)+\mathrm{i}2\pi a)^2}{4(a+\mathrm{i}b)}}\right).$$

Proof. Writing the generalized error function as a difference of two error functions and taking derivative, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}b}\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}},\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)\\&=\frac{\mathrm{d}}{\mathrm{d}b}\operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)-\frac{\mathrm{d}}{\mathrm{d}b}\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}\right).\end{aligned} \tag{3.46}$$

The derivative of the error function is obtained using the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{erf}(f(t))=\frac{2}{\sqrt{\pi}}\mathrm{e}^{-(f(t))^2}\frac{\mathrm{d}}{\mathrm{d}t}f(t).$$

Therefore, we write (3.46) into the following form

$$\frac{\mathrm{d}}{\mathrm{d}b}\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}},\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)$$
$$=\frac{\mathrm{e}^{\frac{((k+2\pi b)-\mathrm{i}2\pi a)^2}{4(a+\mathrm{i}b)}}}{2\sqrt{\pi}}\left(\frac{(k-2\pi b)+\mathrm{i}2\pi a}{(a+\mathrm{i}b)^{\frac{3}{2}}}\right)-\frac{\mathrm{e}^{\frac{((k-2\pi b)+\mathrm{i}2\pi a)^2}{4(a+\mathrm{i}b)}}}{2\sqrt{\pi}}\left(\frac{(k+2\pi b)-\mathrm{i}2\pi a}{(a+\mathrm{i}b)^{\frac{3}{2}}}\right)$$
$$=\frac{\sqrt{a+\mathrm{i}b}(a-\mathrm{i}b)^2}{2\sqrt{\pi}(a^2+b^2)^2}\left(((k-2\pi b)+\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{((k+2\pi b)-\mathrm{i}2\pi a)^2}{4(a+\mathrm{i}b)}}-((k+2\pi b)-\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{((k-2\pi b)+\mathrm{i}2\pi a)^2}{4(a+\mathrm{i}b)}}\right).$$

■

3.6 Asymptotic Expansion of Generalized Error Function

The generalized error functions

$$\operatorname{erf}\left(\frac{\mathrm{i}\,(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}},\frac{\mathrm{i}\,(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)=\operatorname{erf}(z_1,z_2) \tag{3.47}$$

and

$$\operatorname{erf}\left(\frac{\mathrm{i}\,(k+2\pi b)-2\pi a}{2\sqrt{a-\mathrm{i}b}},\frac{\mathrm{i}\,(k-2\pi b)+2\pi a}{2\sqrt{a-\mathrm{i}b}}\right)=\operatorname{erf}(z_3,z_4), \tag{3.48}$$

where

$$
\begin{aligned}
z_1 &= \frac{\mathrm{i}(k-2\pi b)-2a\pi}{2\sqrt{a+\mathrm{i}b}},\ z_2 = \frac{\mathrm{i}(k+2\pi b)+2a\pi}{2\sqrt{a+\mathrm{i}b}} \\
z_3 &= \frac{\mathrm{i}(k+2\pi b)-2a\pi}{2\sqrt{a-\mathrm{i}b}},\ z_4 = \frac{\mathrm{i}(k-2\pi b)+2a\pi}{2\sqrt{a-\mathrm{i}b}}
\end{aligned}
\tag{3.49}
$$

are of relevance in the analysis of chirps. We frequently use such functions to prove inequalities related to fractional Fourier coefficients of exponential linear chirps. The most well-known method for approximating the generalized error functions for large values of the variable is the asymptotic expansion. In this section, we apply the idea of asymptotic expansion given in Section 3.2 to derive the asymptotic expansion of the generalized error functions given in (3.47) and (3.48). First, we show that the complex numbers $z_i,\ i = 1,\ 2,\ 3,\ 4$ given in (3.49) are restricted to some sectors $\theta_0 < \arg z_i < \theta_1,\ i = 1,\ 2,\ 3,\ 4$ of the complex plane. Then we derive the asymptotic expansion of the generalized error functions, by taking $n = 1$ and $n = 2$ respectively and find estimate of the remainder terms.

Lemma 3.6.1. *Let z_1 and z_2 be given in (3.49). There exists $N_0(a,b)$ such that for all $k > N_0(a,b)$, the arguments of z_1 and z_2 lie in the sectors*

$$\frac{\pi}{4} < \arg z_1 < \frac{\pi}{2} \quad \textit{and} \quad \frac{\pi}{4} < \arg z_2 < \frac{\pi}{2}$$

of the complex plane.

Proof. We prove the results for both z_1 and z_2 separately.

The complex number z_1

We shall prove that $\mathrm{Im}(z_1) > \mathrm{Re}(z_1) > 0$ for all $k > N_0(a,b)$. We rationalize the denominator of z_1 to separate real and imaginary part of z_1

$$z_1 = \frac{(\mathrm{i}(k-2\pi b)-2a\pi)\sqrt{a-\mathrm{i}b}}{2\sqrt{a+\mathrm{i}b}\sqrt{a-\mathrm{i}b}}. \tag{3.50}$$

The square root of $a - \mathrm{i}b$ is given by

$$\sqrt{a-\mathrm{i}b} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a} - \frac{\mathrm{i}}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a}. \tag{3.51}$$

We replace (3.51) into (3.50) and simplify to separate z_1 into real and imaginary part

$$
\begin{aligned}
z_1 =& \frac{1}{2\sqrt{2(a^2+b^2)}}\left((k-2\pi b)\sqrt{\sqrt{a^2+b^2}-a} - 2\pi a\sqrt{\sqrt{a^2+b^2}+a}\right) \\
&+ \mathrm{i}\frac{1}{2\sqrt{2(a^2+b^2)}}\left((k-2\pi b)\sqrt{\sqrt{a^2+b^2}+a} + 2\pi a\sqrt{\sqrt{a^2+b^2}-a}\right) \\
=& \mathrm{Re}(z_1) + \mathrm{i}\mathrm{Im}(z_1).
\end{aligned}
\tag{3.52}
$$

From (3.52), we see that there exists $N_0(a,b)$ such that for all $k > N_0(a,b)$, it holds $\mathrm{Im}(z_1) > \mathrm{Re}(z_1) > 0$. Hence, $\frac{\pi}{4} < \arg z_1 < \frac{\pi}{2}$ for all $k > N_0(a,b)$.

The complex number z_2

Similar to z_1, we shall prove that $\mathrm{Im}(z_2) > \mathrm{Re}(z_2) > 0$ for all $k > N_0(a,b)$. We rationalize the denominator of z_2 to get

$$z_2 = \frac{(\mathrm{i}(k+2\pi b)+2a\pi)\sqrt{a-\mathrm{i}b}}{2\sqrt{a+\mathrm{i}b}\sqrt{a-\mathrm{i}b}}. \tag{3.53}$$

Using (3.51) and (3.53), we separate z_2 into real and imaginary part as

$$\begin{aligned} z_2 =& \frac{1}{2\sqrt{2(a^2+b^2)}}\left((k+2\pi b)\sqrt{\sqrt{a^2+b^2}-a}+2\pi a\sqrt{\sqrt{a^2+b^2}+a}\right) \\ &+\mathrm{i}\frac{1}{2\sqrt{2(a^2+b^2)}}\left((k+2\pi b)\sqrt{\sqrt{a^2+b^2}+a}-2\pi a\sqrt{\sqrt{a^2+b^2}-a}\right) \\ =&\mathrm{Re}(z_2)+\mathrm{i}\mathrm{Im}(z_2). \end{aligned} \tag{3.54}$$

From (3.54), we see that there exists $N_0(a,b)$ such that for all $k > N_0(a,b)$, it holds $\mathrm{Im}(z_2) > \mathrm{Re}(z_2) > 0$. Hence, $\frac{\pi}{4} < \arg z_2 < \frac{\pi}{2}$ for all $k > N_0(a,b)$. ■

Lemma 3.6.2. *Let z_3 and z_4 be given in (3.49). There exists $N_0(a,b)$ such that for all $k > N_0(a,b)$, the arguments of z_3 and z_4 lie in the sectors*

$$\frac{\pi}{2} < \arg z_3 < \frac{3\pi}{4} \quad \textit{and} \quad \frac{\pi}{2} < \arg z_4 < \frac{3\pi}{4}$$

of the complex plane.

Proof. We prove the results for both z_3 and z_4 separately.

The complex number z_3

We shall prove that $\mathrm{Re}(z_3) < 0$ and $\mathrm{Im}(z_3) > -\mathrm{Re}(z_3)$ for all $k > N_0(a,b)$. We rationalize the denominator of z_3 to get

$$z_3 = \frac{(\mathrm{i}(k+2\pi b)-2a\pi)\sqrt{a+\mathrm{i}b}}{2\sqrt{a-\mathrm{i}b}\sqrt{a+\mathrm{i}b}}. \tag{3.55}$$

The square root of $a+\mathrm{i}b$ is

$$\sqrt{a+\mathrm{i}b} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a}+\frac{\mathrm{i}}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a}. \tag{3.56}$$

We substitute (3.56) into (3.55) and separate z_3 into real and imaginary part as

$$\begin{aligned} z_3 =& -\frac{1}{2\sqrt{2(a^2+b^2)}}\left((k+2\pi b)\sqrt{\sqrt{a^2+b^2}-a}-2\pi a\sqrt{\sqrt{a^2+b^2}+a}\right) \\ &+\mathrm{i}\frac{1}{2\sqrt{2(a^2+b^2)}}\left((k+2\pi b)\sqrt{\sqrt{a^2+b^2}+a}-2\pi a\sqrt{\sqrt{a^2+b^2}-a}\right) \\ =&\mathrm{Re}(z_3)+\mathrm{i}\mathrm{Im}(z_3). \end{aligned} \tag{3.57}$$

From (3.57), we see that there exists $N_0(a,b)$ such that for all $k > N_0(a,b)$, it holds $\mathrm{Re}(z_3) < 0$ and $\mathrm{Im}(z_3) > -\mathrm{Re}(z_3)$. Hence, $\frac{\pi}{2} < \arg z_3 < \frac{3\pi}{4}$ for all $k > N_0(a,b)$.

The complex number z_4

Similar to z_3, we shall prove that $\mathrm{Re}(z_4) < 0$ and $\mathrm{Im}(z_4) > -\mathrm{Re}(z_4)$ for all $k > N_0(a,b)$. We rationalize the denominator of z_4 to get

$$z_4 = \frac{(\mathrm{i}(k-2\pi b)+2a\pi)\sqrt{a+\mathrm{i}b}}{2\sqrt{a-\mathrm{i}b}\sqrt{a+\mathrm{i}b}}. \tag{3.58}$$

Using (3.56) and (3.58), we separate z_4 into real and imaginary part

$$\begin{aligned} z_4 =& -\frac{1}{2\sqrt{2(a^2+b^2)}}\left((k-2\pi b)\sqrt{\sqrt{a^2+b^2}-a}+2\pi a\sqrt{\sqrt{a^2+b^2}+a}\right) \\ &+\mathrm{i}\frac{1}{2\sqrt{2(a^2+b^2)}}\left((k-2\pi b)\sqrt{\sqrt{a^2+b^2}+a}+2\pi a\sqrt{\sqrt{a^2+b^2}-a}\right) \\ =&\mathrm{Re}(z_4)+\mathrm{i}\mathrm{Im}(z_4). \end{aligned} \tag{3.59}$$

From(3.59), we see that there exists $N_0(a,b)$ such that for all $k > N_0(a,b)$, it holds $\mathrm{Re}(z_4) < 0$ and $\mathrm{Im}(z_4) > -\mathrm{Re}(z_4)$. Hence, $\frac{\pi}{2} < \arg z_4 < \frac{3\pi}{4}$ for all $k > N_0(a,b)$. ■

3.6.1 Asymptotic Expansion for $n = 1$

Lemma 3.6.3. *Let z_1 and z_2 be given in* (3.49). *The generalized error function* (3.47) *has an asymptotic expansion of the form*

$$\begin{aligned} \mathrm{erf}\,(z_1,z_2) = \frac{\mathrm{i}2\sqrt{a+\mathrm{i}b}}{\sqrt{\pi}}\Bigg(&\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2a\pi)} - \frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)} \\ &+\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_1(z_2) - \frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_1(z_1)\Bigg). \end{aligned}$$

The error terms $\varepsilon_1(z_1)$ and $\varepsilon_1(z_2)$ have the following bounds

$$|\varepsilon_1(z_1)| \le \frac{2\sqrt{a^2+b^2}}{((k-2\pi b)^2+4\pi^2a^2)} \quad \text{and} \quad |\varepsilon_1(z_2)| \le \frac{2\sqrt{a^2+b^2}}{((k+2\pi b)^2+4\pi^2a^2)}.$$

Proof. The generalized error function can be written as a difference of two complementary error functions

$$\mathrm{erf}(z_1,z_2) = \mathrm{erfc}(z_1) - \mathrm{erfc}(z_2). \tag{3.60}$$

For a complex number z, the asymptotic expansion of a complementary error function provides the best approximation for some $n = N_0(z)$. We set $N_0(z) = 1$ in Lemma 3.2.1.

The asymptotic expansion of the generalized error function (3.60) reads as

$$\begin{aligned}\operatorname{erf}(z_1,z_2) =&\frac{2\sqrt{a+\mathrm{i}b}\mathrm{e}^{-z_1^2}}{\sqrt{\pi}\,(\mathrm{i}(k-2\pi b)-2\pi a)}+\frac{2\sqrt{a+\mathrm{i}b}\mathrm{e}^{-z_1^2}}{\sqrt{\pi}\,(\mathrm{i}(k-2\pi b)-2\pi a)}\varepsilon_1(z_1)\\ &-\frac{2\sqrt{a+\mathrm{i}b}\mathrm{e}^{-z_2^2}}{\sqrt{\pi}\,(\mathrm{i}(k+2\pi b)+2\pi a)}-\frac{2\sqrt{a+\mathrm{i}b}\mathrm{e}^{-z_2^2}}{\sqrt{\pi}\,(\mathrm{i}(k+2\pi b)+2\pi a)}\varepsilon_1(z_2)\\ =&\frac{\mathrm{i}2\sqrt{a+\mathrm{i}b}}{\sqrt{\pi}}\left(\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2a\pi)}-\frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\right.\\ &\left.+\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_1(z_2)-\frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_1(z_1)\right).\end{aligned}$$

We have proved in Lemma 3.6.1 that for all $k > N_0(a,b)$, the arguments of z_1 and z_2 lie in the sectors

$$\frac{\pi}{4} < \arg z_1 < \frac{\pi}{2} \quad \text{and} \quad \frac{\pi}{4} < \arg z_2 < \frac{\pi}{2}$$

of the complex plane respectively. From Lemma 3.2.1, we have the following estimates of the remainders $\varepsilon_1(z_1)$ and $\varepsilon_1(z_2)$

$$|\varepsilon_1(z_1)| \le \frac{2\sqrt{a^2+b^2}}{((k-2\pi b)^2+4\pi^2a^2)} \quad \text{and} \quad |\varepsilon_1(z_2)| \le \frac{2\sqrt{a^2+b^2}}{((k+2\pi b)^2+4\pi^2a^2)}.$$

■

Lemma 3.6.4. *Let z_3 and z_4 be given in* (3.49). *Then the generalized error function* (3.48) *has the following asymptotic expansion*

$$\begin{aligned}\operatorname{erf}(z_1,z_2) = \frac{-\mathrm{i}2\sqrt{a-\mathrm{i}b}}{\sqrt{\pi}}&\left(\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)^3}-\frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)^3}\right.\\ &\left.+\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_1(z_3)-\frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_1(z_4)\right).\end{aligned}$$

The estimates of the remainder terms $\varepsilon_1(z_3)$ and $\varepsilon_1(z_4)$ are

$$|\varepsilon_1(z_3)| < \frac{2\sqrt{2(a^2+b^2)}}{((k+2\pi b)^2+4\pi^2a^2)} \quad \textit{and} \quad |\varepsilon_1(z_4)| < \frac{2\sqrt{2(a^2+b^2)}}{((k-2\pi b)^2+4\pi^2a^2)}.$$

Proof. Analogous to Lemma 3.6.3, the asymptotic expansion of the generalized error function (3.48) is given by

$$\begin{aligned}\operatorname{erf}(z_3,z_4) = \frac{-\mathrm{i}2\sqrt{a-\mathrm{i}b}}{\sqrt{\pi}}&\left(\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)^3}-\frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)^3}\right.\\ &\left.+\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_1(z_3)-\frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_1(z_4)\right).\end{aligned}$$

We have proved in Lemma 3.6.2 that for all $k > N_0(a,b)$, the arguments of z_3 and z_4 lie in the sectors

$$\frac{\pi}{2} < \arg z_3 < \frac{3\pi}{4} \quad \text{and} \quad \frac{\pi}{2} < \arg z_4 < \frac{3\pi}{4}$$

of the complex plane respectively. From Lemma 3.2.1, we have the following estimates of the remainders $\varepsilon_1(z_3)$ and $\varepsilon_1(z_4)$

$$\begin{aligned} |\varepsilon_1(z_3)| &\leq \frac{2\sqrt{a^2+b^2}}{((k+2\pi b)^2+4\pi^2a^2)} \csc(\arg z_3) \\ |\varepsilon_1(z_4)| &\leq \frac{2\sqrt{a^2+b^2}}{((k-2\pi b)^2+4\pi^2a^2)} \csc(\arg z_4). \end{aligned} \tag{3.61}$$

Since $\frac{\pi}{2} \leq \arg z_3 < \frac{3\pi}{4}$, we have

$$\frac{1}{\sqrt{2}} < \sin(\arg z_3) \leq 1.$$

Consequently, the bounds for $\csc(\arg z_3)$ are

$$1 \leq \csc(\arg z_3) < \sqrt{2}.$$

Similarly, we obtain

$$1 \leq \csc(\arg z_4) < \sqrt{2}.$$

Hence, the estimates of $\varepsilon_1(z_3)$ and $\varepsilon_1(z_4)$ in (3.61) becomes

$$|\varepsilon_1(z_3)| < \frac{2\sqrt{2(a^2+b^2)}}{((k+2\pi b)^2+4\pi^2a^2)} \quad \text{and} \quad |\varepsilon_1(z_4)| < \frac{2\sqrt{2(a^2+b^2)}}{((k-2\pi b)^2+4\pi^2a^2)}.$$

■

3.6.2 Asymptotic Expansion for $n = 2$

Lemma 3.6.5. *Let z_1 and z_2 be given in (3.49). The generalized error function (3.47) has an asymptotic expansion of the form*

$$\begin{aligned} \operatorname{erf}(z_1, z_2) = \frac{\mathrm{i}2\sqrt{a+\mathrm{i}b}}{\sqrt{\pi}} \Bigg(& \frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2a\pi)^3} \left(((k+2\pi b)-\mathrm{i}2\pi a)^2 + 2(a+\mathrm{i}b)\right) \\ & - \frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)^3} \left(((k-2\pi b)+\mathrm{i}2\pi a)^2 + 2(a+\mathrm{i}b)\right) \\ & + \frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2\pi a)} \varepsilon_2(z_2) - \frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)} \varepsilon_2(z_1) \Bigg), \end{aligned}$$

where the error terms $\varepsilon_2(z_1)$ and $\varepsilon_2(z_2)$ have the following bounds

$$|\varepsilon_2(z_1)| \leq \frac{12(a^2+b^2)}{((k-2\pi b)^2+4\pi^2a^2)^2} \quad \text{and} \quad |\varepsilon_2(z_2)| \leq \frac{12(a^2+b^2)}{((k+2\pi b)^2+4\pi^2a^2)^2}.$$

Proof. The generalized error function can be written as a difference of two complementary error functions

$$\operatorname{erf}(z_1, z_2) = \operatorname{erfc}(z_1) - \operatorname{erfc}(z_2). \tag{3.62}$$

For a complex number z, the asymptotic expansion of a complementary error function provides the best approximation for some $n = N_0(z)$. We set $N_0(z) = 2$ in Lemma 3.2.1. The asymptotic expansion of the generalized error function (3.62) reads as

$$\begin{aligned}
\operatorname{erf}(z_1, z_2) =& \frac{2\sqrt{a+\mathrm{i}b}\mathrm{e}^{-z_1^2}}{\sqrt{\pi}\left(\mathrm{i}(k-2\pi b)-2\pi a\right)}\left(1-\frac{4(a+\mathrm{i}b)}{2\left(\mathrm{i}(k-2\pi b)-2\pi a\right)^2}+\varepsilon_2(z_1)\right) \\
&-\frac{2\sqrt{a+\mathrm{i}b}\mathrm{e}^{-z_2^2}}{\sqrt{\pi}\left(\mathrm{i}(k+2\pi b)+2\pi a\right)}\left(1-\frac{4(a+\mathrm{i}b)}{2\left(\mathrm{i}(k+2\pi b)+2\pi a\right)^2}+\varepsilon_2(z_2)\right) \\
=& \frac{\mathrm{i}2\sqrt{a+\mathrm{i}b}}{\sqrt{\pi}}\left(\frac{\mathrm{e}^{-z_2^2}}{\left((k+2\pi b)-\mathrm{i}2a\pi\right)^3}\left(\left((k+2\pi b)-\mathrm{i}2\pi a\right)^2+2(a+\mathrm{i}b)\right)\right. \\
&-\frac{\mathrm{e}^{-z_1^2}}{\left((k-2\pi b)+\mathrm{i}2\pi a\right)^3}\left(\left((k-2\pi b)+\mathrm{i}2\pi a\right)^2+2(a+\mathrm{i}b)\right) \\
&\left.+\frac{\mathrm{e}^{-z_2^2}}{\left((k+2\pi b)-\mathrm{i}2\pi a\right)}\varepsilon_2(z_2)-\frac{\mathrm{e}^{-z_1^2}}{\left((k-2\pi b)+\mathrm{i}2\pi a\right)}\varepsilon_2(z_1)\right).
\end{aligned}$$

We have proved in Lemma 3.6.1 that for all $k > N_0(a, b)$, the arguments of z_1 and z_2 lie in the sectors

$$\frac{\pi}{4} < \arg z_1 < \frac{\pi}{2} \quad \text{and} \quad \frac{\pi}{4} < \arg z_2 < \frac{\pi}{2}$$

of the complex plane respectively. From Lemma 3.2.1, we use the following estimates of the remainders $\varepsilon_2(z_1)$ and $\varepsilon_2(z_2)$

$$|\varepsilon_2(z_1)| \le \frac{12(a^2+b^2)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^2} \quad \text{and} \quad |\varepsilon_2(z_2)| \le \frac{12(a^2+b^2)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^2}.$$

∎

Lemma 3.6.6. *Let z_3 and z_4 be given in* (3.49). *Then the generalized error function* (3.48) *has the following asymptotic expansion*

$$\begin{aligned}
\operatorname{erf}(z_3, z_4) = \frac{-\mathrm{i}2\sqrt{a-\mathrm{i}b}}{\sqrt{\pi}}&\left(\frac{\mathrm{e}^{-z_3^2}}{\left((k+2\pi b)+\mathrm{i}2\pi a\right)^3}\left(\left((k+2\pi b)+\mathrm{i}2\pi a\right)^2+2(a-\mathrm{i}b)\right)\right. \\
&-\frac{\mathrm{e}^{-z_4^2}}{\left((k-2\pi b)-\mathrm{i}2\pi a\right)^3}\left(\left((k-2\pi b)-\mathrm{i}2\pi a\right)^2+2(a-\mathrm{i}b)\right) \\
&\left.+\frac{\mathrm{e}^{-z_3^2}}{\left((k+2\pi b)+\mathrm{i}2\pi a\right)}\varepsilon_2(z_3)-\frac{\mathrm{e}^{-z_4^2}}{\left((k-2\pi b)-\mathrm{i}2\pi a\right)}\varepsilon_2(z_4)\right),
\end{aligned}$$

where the estimates of the remainder terms $\varepsilon_2(z_3)$ *and* $\varepsilon_2(z_4)$ *are*

$$|\varepsilon_2(z_3)| < \frac{12\sqrt{2}(a^2+b^2)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^2} \quad \textit{and} \quad |\varepsilon_2(z_4)| < \frac{12\sqrt{2}(a^2+b^2)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^2}.$$

Proof. Analogous to Lemma 3.6.5, the asymptotic expansion of the generalized error function (3.48) is given by

$$\begin{aligned}\operatorname{erf}(z_3,z_4) = \frac{-\mathrm{i}2\sqrt{a-\mathrm{i}b}}{\sqrt{\pi}}\Bigg(& \frac{\mathrm{e}^{-z_3^2}}{\left((k+2\pi b)+\mathrm{i}2\pi a\right)^3}\left(\left((k+2\pi b)+\mathrm{i}2\pi a\right)^2+2(a-\mathrm{i}b)\right) \\ & -\frac{\mathrm{e}^{-z_4^2}}{\left((k-2\pi b)-\mathrm{i}2\pi a\right)^3}\left(\left((k-2\pi b)-\mathrm{i}2\pi a\right)^2+2(a-\mathrm{i}b)\right) \\ & +\frac{\mathrm{e}^{-z_3^2}}{\left((k+2\pi b)+\mathrm{i}2\pi a\right)}\varepsilon_2(z_3)-\frac{\mathrm{e}^{-z_4^2}}{\left((k-2\pi b)-\mathrm{i}2\pi a\right)}\varepsilon_2(z_4)\Bigg).\end{aligned}$$

We have proved in Lemma 3.6.2 that for all $k > N_0(a,b)$, the arguments of z_3 and z_4 lie in the sectors

$$\frac{\pi}{2} < \arg z_3 < \frac{3\pi}{4} \quad \text{and} \quad \frac{\pi}{2} < \arg z_4 < \frac{3\pi}{4}$$

of the complex plane respectively. From Lemma 3.2.1, we have the following estimates of the remainders $\varepsilon_2(z_3)$ and $\varepsilon_2(z_4)$

$$\begin{aligned}|\varepsilon_2(z_3)| &\le \frac{12(a^2+b^2)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^2}\csc(\arg z_3) \\ |\varepsilon_2(z_4)| &\le \frac{12(a^2+b^2)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^2}\csc(\arg z_4).\end{aligned} \tag{3.63}$$

From Lemma 3.6.4, we have

$$1 \le \csc(\arg z_3) < \sqrt{2} \quad \text{and} \quad 1 \le \csc(\arg z_4) < \sqrt{2}.$$

Hence, the estimates of $\varepsilon_2(z_3)$ and $\varepsilon_2(z_4)$ in (3.63) becomes

$$|\varepsilon_2(z_3)| < \frac{12\sqrt{2}(a^2+b^2)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^2} \quad \text{and} \quad |\varepsilon_2(z_4)| < \frac{12\sqrt{2}(a^2+b^2)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^2}.$$

∎

Chapter 4

Fractional Fourier Series Approximation of Exponential Linear Chirps

The subject of this main chapter is to find the optimal basis in FrFS expansion of exponential linear chirps. As described in Chapter 1, FrFS expansion with an optimal parameter has an advantage over the classical Fourier series for the approximation of exponential linear chirps. The objective of this chapter is to confirm such an advantage through calculus and asymptotic expansions. The concept of FrFS expansion of chirps was first considered by Pei et al. [69] where the analysis of linear chirps with amplitude function $A(t) = 1$ was carried out. The FrFS expansion of the chirps with amplitude function $A(t)$ as the real quadratic exponential was demonstrated numerically by Yu [30] and Cöetmellec et al. [68]. In contrast to the approach used in [30, 68, 69], we study FrFS expansion of linear chirps on $[-\pi, \pi]$ from mathematical viewpoint.

This chapter is mainly divided into four parts. In Section 4.1, we derive an expression for the absolute value of fractional Fourier coefficients of exponential linear chirps for all $k \in \mathbb{Z}$ and deduce an expression for $k = 0$. Section 4.2 is devoted to optimality properties of fractional Fourier coefficients. We prove the maximality property of fractional Fourier coefficients of zero degree and the minimality property of fractional Fourier coefficients of large degree. Further, we find decay properties of fractional Fourier coefficients of exponential linear chirps. Finally, we prove the best approximation property of FrFS of exponential linear chirps for large $N \in \mathbb{N}_0$. In Section 4.3, we restrict our research to a domain $\mathcal{D}$ for which the fractional Fourier coefficients are monotone. We prove the monotonicity properties of fractional Fourier coefficients of zero degree, fractional Fourier coefficients of large degree and L^2 error. In the last part, we draw conclusions and give directions for future research.

4.1 Expression for the Fractional Fourier Coefficients

In this section, we derive an expression for the absolute value of the fractional Fourier coefficients $\hat{f}_\alpha(k)$ for all $k \in \mathbb{Z}$. Further, we deduce the expression for $\left|\hat{f}_\alpha(0)\right|$.

Lemma 4.1.1. *For an exponential linear chirp*

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi],$$

defined for given $\gamma > 0$ and $\mu \in \mathbb{R}^+$, we have

$$\begin{aligned}\left|\hat{f}_\alpha(k)\right|^2 =& \frac{\mathrm{e}^{\frac{-\pi\gamma k^2}{2\left(\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2\right)}}}{16\pi\sqrt{\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2}} \\ &\times \left|\operatorname{erf}\left(\frac{\mathrm{i}\,(k-\pi(2\pi\mu-\cot\alpha))-2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}, \frac{\mathrm{i}\,(k+\pi(2\pi\mu-\cot\alpha))+2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}\right)\right|^2.\end{aligned}$$

Proof. By definition, the fractional Fourier coefficients are given by

$$\hat{f}_\alpha(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\gamma^\mu(u)\overline{\phi_{k,\alpha}(u)}\,\mathrm{d}u. \tag{4.1}$$

Taking absolute value of $\hat{f}_\alpha(k)$ and substituting the values of $f_\gamma^\mu(u)$ and $\overline{\phi_{k,\alpha}(u)}$ into (4.1) gives

$$\begin{aligned}\left|\hat{f}_\alpha(k)\right|^2 &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)u^2}\mathrm{e}^{\frac{\mathrm{i}}{2}u^2\cot\alpha}\mathrm{e}^{-\mathrm{i}ku}\,\mathrm{d}u\right|^2 \\ &= \frac{1}{4\pi^2}\left|\int_{-\pi}^{\pi} \mathrm{e}^{-\pi\gamma u^2}\mathrm{e}^{\frac{-\mathrm{i}}{2}(2\pi\mu-\cot\alpha)u^2}\mathrm{e}^{-\mathrm{i}ku}\,\mathrm{d}u\right|^2.\end{aligned} \tag{4.2}$$

Using Lemma 3.5.1, the integral (4.2) is represented in terms of a generalized error function

$$\begin{aligned}\left|\hat{f}_\alpha(k)\right|^2 =& \frac{1}{4\pi^2}\left|\frac{\sqrt{\pi}\mathrm{e}^{\frac{-k^2}{4\left(\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)\right)}}}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}\right|^2 \\ &\times \left|\operatorname{erf}\left(\frac{\mathrm{i}\,(k-\pi(2\pi\mu-\cot\alpha))-2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}, \frac{\mathrm{i}\,(k+\pi(2\pi\mu-\cot\alpha))+2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}\right)\right|^2.\end{aligned}$$

Simplification of absolute value gives

$$\left|\hat{f}_\alpha(k)\right|^2 = \frac{\mathrm{e}^{\frac{-\pi\gamma k^2}{2\left(\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2\right)}}}{16\pi\sqrt{\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2}} \times \left|\operatorname{erf}\left(\frac{\mathrm{i}\,(k-\pi(2\pi\mu-\cot\alpha))-2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}, \frac{\mathrm{i}\,(k+\pi(2\pi\mu-\cot\alpha))+2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}\right)\right|^2. \tag{4.3}$$

■

This is a general expression of fractional Fourier coefficients of exponential linear chirps. In (4.3), we see that the fractional Fourier coefficients of exponential linear chirps depend on the fractional parameter α and fractional Fourier degree index k. Fractional Fourier coefficients of zero degree are obtained from the following corollary.

Corollary. *The expression for* $\left|\hat{f}_\alpha(0)\right|$ *is given by*

$$\left|\hat{f}_\alpha(0)\right|^2 = \frac{1}{4\pi\sqrt{\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2}}\left|\operatorname{erf}\left(\pi\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}\right)\right|^2.$$

Proof. From (4.3), we find an expression for $\left|\hat{f}_\alpha(0)\right|$ as

$$\left|\hat{f}_\alpha(0)\right|^2 = \frac{1}{16\pi\sqrt{\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2}} \times \left|\operatorname{erf}\left(\frac{-\mathrm{i}\pi(2\pi\mu-\cot\alpha)-2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}, \frac{\mathrm{i}\pi(2\pi\mu-\cot\alpha)+2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}\right)\right|^2.$$

Using the property

$$\operatorname{erf}(-z,z) = 2\operatorname{erf}(z),$$

we have

$$\left|\hat{f}_\alpha(0)\right|^2 = \frac{1}{4\pi\sqrt{\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2}}\left|\operatorname{erf}\left(\pi\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}\right)\right|^2. \tag{4.4}$$

■

4.2 Optimality Properties of Fractional Fourier Coefficients

This section aims to demonstrate that the FrFS expansions and their theory can be successfully applied to the linear chirps with the amplitude function as a real quadratic exponential. Such expansions with an optimal parameter give the fastest possible convergence rate among all parameters in the fractional Fourier parameter domain. We prove optimalilty properties of FrFS which includes the maximality property of fractional Fourier coefficients of zero degree and the minimality property of fractional Fourier coefficients of large degree. Finally, the best approximation of the FrFS is proved with the optimal fractional Fourier basis functions for large $N \in \mathbb{N}_0$.

4.2.1 The Maximality Property of Fractional Fourier Coefficients

The central idea behind the use of optimal fractional Fourier basis is to utilize a few fractional Fourier coefficients for the reconstruction of chirps. In other words, a few fractional Fourier coefficients carry the vital information necessary to reconstruct the chirps. By adjusting the fractional Fourier parameter to α^*, fractional Fourier coefficients are compressed in the fractional Fourier parameter domain; the fractional Fourier coefficients with significant magnitudes are located in a spectrum. In this case, it is expected that the power of the coefficients is concentrated on just a few coefficients. The fractional Fourier coefficient of zero degree is the main coefficient which possesses the maximum value among all fractional Fourier coefficients. We prove the following theorem.

Theorem 4.2.1. *Let*

$$f(t) = f_{\gamma}^{\mu}(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ *and* $\mu \in \mathbb{R}^+$. *Then for all* α *with* $0 < \alpha \leq \frac{\pi}{2}$ *and*

$$\alpha \neq \alpha^* = \arctan\left(\frac{1}{2\pi\mu}\right),$$

we have

$$\left|\hat{f}_{\alpha^*}(0)\right| > \left|\hat{f}_{\alpha}(0)\right|.$$

Proof. We divide the fractional Fourier parameter domain into two parts namely $\alpha > \alpha^*$ and $\alpha < \alpha^*$.

For $\alpha > \alpha^*$

Let $J_0^R(\alpha)$ be a function given by

$$J_0^R(\alpha) = \left|\hat{f}_{\alpha^*}(0)\right|^2 - \left|\hat{f}_\alpha(0)\right|^2. \tag{4.5}$$

From (4.4), we have the expression

$$\left|\hat{f}_\alpha(0)\right|^2 = \frac{1}{4\pi\sqrt{\pi^2\gamma^2 + \frac{1}{4}(2\pi\mu - \cot\alpha)^2}} \left|\mathrm{erf}\left(\pi\sqrt{\pi\gamma + \frac{\mathrm{i}}{2}(2\pi\mu - \cot\alpha)}\right)\right|^2. \tag{4.6}$$

For optimal parameter, we have the relation $\alpha = \alpha^* = \arctan\left(\frac{1}{2\pi\mu}\right)$ which leads to $\frac{1}{2}\left(2\pi\mu - \cot\alpha\right) = 0$. Therefore, (4.6) reduces to

$$\left|\hat{f}_{\alpha^*}(0)\right|^2 = \frac{1}{4\pi^2\gamma}\left|\mathrm{erf}\left(\pi\sqrt{\pi\gamma}\right)\right|^2. \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5), we write $J_0^R(\alpha)$ as

$$\begin{aligned} J_0^R(\alpha) =& \frac{1}{4\pi^2\gamma}\left|\mathrm{erf}\left(\pi\sqrt{\pi\gamma}\right)\right|^2 \\ & - \frac{1}{4\pi\sqrt{\pi^2\gamma^2 + \frac{1}{4}(2\pi\mu - \cot\alpha)^2}} \left|\mathrm{erf}\left(\pi\sqrt{\pi\gamma + \frac{\mathrm{i}}{2}(2\pi\mu - \cot\alpha)}\right)\right|^2. \end{aligned} \tag{4.8}$$

For given $\gamma > 0$ and $\mu \in \mathbb{R}^+$, we shall show that $J_0^R(\alpha) > 0$ on $\alpha^* < \alpha \leq \frac{\pi}{2}$. We simplify (4.8) using the substitution

$$\pi\gamma = a(\gamma) \quad \text{and} \quad \frac{1}{2}(2\pi\mu - \cot\alpha) = b(\alpha). \tag{4.9}$$

For the sake of convenience, we write a and b instead of $a(\gamma)$ and $b(\alpha)$. When $\alpha^* < \alpha \leq \frac{\pi}{2}$, then $a,\ b > 0$ and $J_0^R(\alpha)$ in (4.8) becomes $\Lambda^R(b)$, where

$$\Lambda^R(b) = \frac{1}{4\pi a}\left|\mathrm{erf}\left(\pi\sqrt{a}\right)\right|^2 - \frac{1}{4\pi\sqrt{a^2+b^2}}\left|\mathrm{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\right|^2.$$

We equivalently prove that $\Lambda^R(b) > 0$ for all $a,\ b > 0$. We write $\Lambda^R(b)$ as

$$\Lambda^R(b) = \frac{\pi}{4}\left(\frac{\mathrm{erf}\left(\pi\sqrt{a}\right)}{\pi\sqrt{a}}\right)^2 - \frac{\pi}{4}\left|\frac{\mathrm{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)}{\pi\sqrt{a+\mathrm{i}b}}\right|^2.$$

Using Lemma 3.3.1, we write error functions in terms of integrals on the unit interval

$$\begin{aligned} \Lambda^R(b) =& \frac{\pi}{4}\left(\frac{2}{\sqrt{\pi}}\int_0^1 \mathrm{e}^{-a\pi^2u^2}\,\mathrm{d}u\right)^2 \\ & - \frac{\pi}{4}\left[\left(\frac{2}{\sqrt{\pi}}\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\mathrm{d}u\right)^2 + \left(\frac{2}{\sqrt{\pi}}\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\mathrm{d}u\right)^2\right]. \end{aligned}$$

We represent the square of integrals in the form of double integrals on the unit square as

$$\Lambda^R(b) = \int_0^1 \int_0^1 \mathrm{e}^{-a\pi^2(u^2+t^2)}\ \mathrm{d}u\mathrm{d}t - \left(\int_0^1 \int_0^1 \mathrm{e}^{-a\pi^2(u^2+t^2)} \cos\left(b\pi^2 u^2\right) \cos\left(b\pi^2 t^2\right) \mathrm{d}u\mathrm{d}t \right.$$
$$\left. + \int_0^1 \int_0^1 \mathrm{e}^{-a\pi^2(u^2+t^2)} \sin\left(b\pi^2 u^2\right) \sin\left(b\pi^2 t^2\right) \mathrm{d}u\mathrm{d}t \right).$$

We write as a single integral form

$$\Lambda^R(b) = \int_0^1 \int_0^1 \mathrm{e}^{-a\pi^2(u^2+t^2)} \left(1 - \cos\left(b\pi^2(u^2 - t^2)\right)\right)\ \mathrm{d}u\mathrm{d}t. \tag{4.10}$$

In (4.10), the integrand satisfies

$$\mathrm{e}^{-a\pi^2(u^2+t^2)} \left(1 - \cos\left(b\pi^2(u^2 - t^2)\right)\right) > 0$$

for all $a,\ b > 0$ on the unit square. Therefore $\Lambda^R(b) > 0$.

For $\alpha < \alpha^*$

Let $J_0^L(\alpha)$ be a function given by

$$J_0^L(\alpha) = \left|\hat{f}_\alpha(0)\right|^2 - \left|\hat{f}_{\alpha^*}(0)\right|^2 = -\left(\left|\hat{f}_{\alpha^*}(0)\right|^2 - \left|\hat{f}_\alpha(0)\right|^2\right). \tag{4.11}$$

Substituting (4.6) and (4.7) into (4.11), we have

$$J_0^L(\alpha) = -\left(\frac{1}{4\pi^2\gamma} \left|\operatorname{erf}\left(\pi\sqrt{\pi\gamma}\right)\right|^2 \right.$$
$$\left. - \frac{1}{4\pi\sqrt{\pi^2\gamma^2 + \frac{1}{4}(2\pi\mu - \cot\alpha)^2}} \left|\operatorname{erf}\left(\pi\sqrt{\pi\gamma + \frac{\mathrm{i}}{2}(2\pi\mu - \cot\alpha)}\right)\right|^2\right). \tag{4.12}$$

For given $\gamma > 0$ and $\mu \in \mathbb{R}^+$, we shall show that $J_0^L(\alpha) < 0$ on $0 < \alpha < \alpha^*$. We simplify (4.12) using the substitution given in (4.9). When $0 < \alpha < \alpha^*$, then $a > 0$ and $b < 0$. The function $J_0^L(\alpha)$ in (4.12) becomes $\Lambda^L(b)$, where

$$\Lambda^L(b) = -\left(\frac{1}{4\pi a}\left|\operatorname{erf}\left(\pi\sqrt{a}\right)\right|^2 - \frac{1}{4\pi\sqrt{a^2+b^2}}\left|\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\right|^2\right).$$

We equivalently prove that $\Lambda^L(b) < 0$ for all $a > 0$ and $b < 0$. Similar calculations as in case $\alpha > \alpha^*$ yield

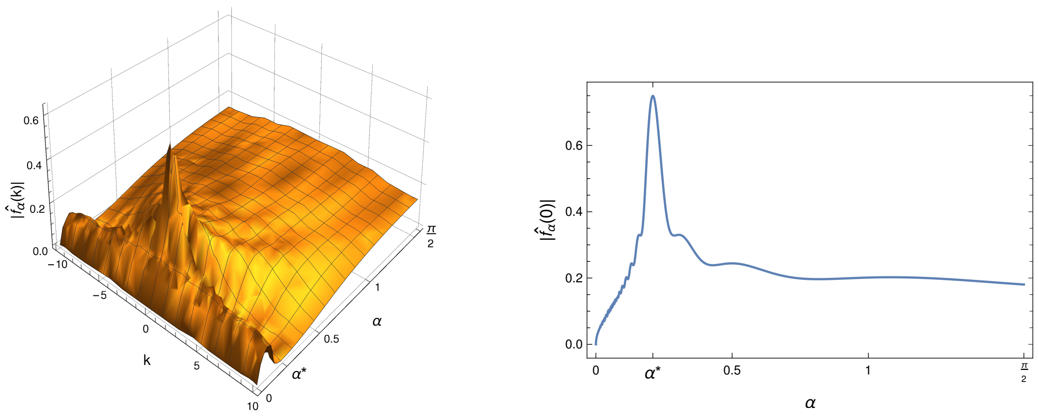
$$\Lambda^L(b) = -\int_0^1 \int_0^1 \mathrm{e}^{-a\pi^2(u^2+t^2)} \left(1 - \cos\left(b\pi^2(u^2 - t^2)\right)\right)\ \mathrm{d}u\mathrm{d}t. \tag{4.13}$$

In (4.13), the integrand satisfies

$$\mathrm{e}^{-a\pi^2(u^2+t^2)} \left(1 - \cos\left(b\pi^2(u^2 - t^2)\right)\right) > 0$$

for all $a > 0$ and $b < 0$ on the unit square. Hence, $\Lambda^L(b) < 0$. ■

We have proved in Theorem 4.2.1 that fractional Fourier coefficients of zero degree have the maximum magnitude when $\alpha = \alpha^*$. The optimal behavior of fractional Fourier coefficients is illustrated in Figure 4.1.



(a) (b)

Figure 4.1: For the chirp $f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}$ with $\gamma = \frac{1}{10\pi}$ and $\mu = \frac{3}{4}$, (a) the three-dimensional plot of $\left|\hat{f}_\alpha(k)\right|$ for $-10 \le k \le 10$ on fractional Fourier parameter domain $0 < \alpha \le \frac{\pi}{2}$, and (b) the graph of $\left|\hat{f}_\alpha(0)\right|$ on $0 < \alpha \le \frac{\pi}{2}$.

In Figure 4.1, we observe that for given $\gamma > 0$ and $\mu \in \mathbb{R}^+$, the absolute value of the fractional Fourier coefficients $\left|\hat{f}_{\alpha^*}(0)\right|$ have the maximum value. Therefore, the optimal parameter α^* provides the most localized fractional Fourier coefficients in the fractional Fourier parameter domain $0 < \alpha \le \frac{\pi}{2}$. The optimal parameter α^* that makes the absolute value of fractional Fourier coefficient $\left|\hat{f}_{\alpha^*}(0)\right|$ into a peak.

4.2.2 The Minimality Property of Fractional Fourier Coefficients

By choosing fractional Fourier parameter to be α^*, fractional Fourier coefficients are concentrated around $k = 0$. Therefore, fractional Fourier coefficients attain large values over a small set of points. Consequently, fractional Fourier coefficients of large degree contain relatively small values at all points and go to zero rapidly. In Theorem 4.2.1, we have proved that fractional Fourier coefficients of small degree are the most localized when the fractional Fourier parameter chosen to be α^*. Now we show that there exists $N(\gamma, \mu) \in \mathbb{N}_0$ such that for all $|k| > N(\gamma, \mu)$, the fractional Fourier coefficients have the minimum value when $\alpha = \alpha^*$.

Theorem 4.2.2. *Let*

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ *and* $\mu \in \mathbb{R}^+$. *Then there exists* $N(\gamma, \mu) \in \mathbb{N}_0$ *such that for all* $|k| > N(\gamma, \mu)$, *we have*

$$\left|\hat{f}_{\alpha^*}(k)\right| < \left|\hat{f}_\alpha(k)\right|$$

on $0 < \alpha \leq \frac{\pi}{2}$ *when* $\alpha \neq \alpha^*$.

Proof. We divide the fractional Fourier parameter domain into two parts namely $\alpha > \alpha^*$ and $\alpha < \alpha^*$.

For $\alpha > \alpha^*$

Let $E_k^R(\alpha)$ be a function given by

$$E_k^R(\alpha) = \left|\hat{f}_{\alpha^*}(k)\right|^2 - \left|\hat{f}_\alpha(k)\right|^2. \tag{4.14}$$

From Lemma 4.1.1, we have

$$\begin{aligned}\left|\hat{f}_\alpha(k)\right|^2 =& \frac{\mathrm{e}^{\frac{-\pi\gamma k^2}{2\left(\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2\right)}}}{16\pi\sqrt{\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2}} \\ &\times \left|\mathrm{erf}\left(\frac{\mathrm{i}\,(k-\pi(2\pi\mu-\cot\alpha))-2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}, \frac{\mathrm{i}\,(k+\pi(2\pi\mu-\cot\alpha))+2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}\right)\right|^2.\end{aligned} \tag{4.15}$$

For optimal parameter, we have the relation $\alpha = \alpha^* = \arctan\left(\frac{1}{2\pi\mu}\right)$ which leads to $\frac{1}{2}(\cot\alpha - 2\pi\mu) = 0$. Then (4.15) reduces to

$$\left|\hat{f}_\alpha(k)\right|^2 = \frac{\mathrm{e}^{\frac{-k^2}{2\pi\gamma}}}{16\pi^2\gamma}\left|\mathrm{erf}\left(\frac{\mathrm{i}k-2\pi^2\gamma}{2\sqrt{\pi\gamma}}, \frac{\mathrm{i}k+2\pi^2\gamma}{2\sqrt{\pi\gamma}}\right)\right|^2. \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.14), we have

$$\begin{aligned}E_k^R(\alpha) =& \frac{\mathrm{e}^{\frac{-k^2}{2\pi\gamma}}}{16\pi^2\gamma}\left|\mathrm{erf}\left(\frac{\mathrm{i}k-2\pi^2\gamma}{2\sqrt{\pi\gamma}}, \frac{\mathrm{i}k+2\pi^2\gamma}{2\sqrt{\pi\gamma}}\right)\right|^2 - \frac{\mathrm{e}^{\frac{-\pi\gamma k^2}{2\left(\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2\right)}}}{16\pi\sqrt{\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2}} \\ &\times \left|\mathrm{erf}\left(\frac{\mathrm{i}\,(k-\pi(2\pi\mu-\cot\alpha))-2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}, \frac{\mathrm{i}\,(k+\pi(2\pi\mu-\cot\alpha))+2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}\right)\right|^2.\end{aligned}$$

For given $\gamma > 0$ and $\mu \in \mathbb{R}^+$, we shall show that $E_k^R(\alpha) < 0$ when $|k| > N(\gamma, \mu)$. We simplify the above expression using the substitution

$$\pi\gamma = a(\gamma) \quad \text{and} \quad \frac{1}{2}(2\pi\mu - \cot\alpha) = b(\alpha). \tag{4.17}$$

For the sake of convenience, we write a and b instead of $a(\gamma)$ and $b(\alpha)$. When $\alpha^* < \alpha \leq \frac{\pi}{2}$, then $a,\ b > 0$ and $E_k^R(\alpha)$ becomes $W_k^R(b)$, given by

$$\begin{aligned} W_k^R(b) =& \frac{\mathrm{e}^{\frac{-k^2}{2a}}}{16\pi a}\left|\operatorname{erf}\left(\frac{\mathrm{i}k - 2\pi a}{2\sqrt{a}}, \frac{\mathrm{i}k + 2\pi a}{2\sqrt{a}}\right)\right|^2 \\ &- \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{16\pi\sqrt{a^2+b^2}}\left|\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)\right|^2 \\ =& \left|\hat{f}_0(k)\right|^2 - \left|\hat{f}_b(k)\right|^2, \end{aligned} \tag{4.18}$$

where

$$\left|\hat{f}_b(k)\right|^2 = \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{16\pi\sqrt{a^2+b^2}}\left|\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)\right|^2.$$

We equivalently prove that $W_k^R(b) < 0$ for all $a,\ b > 0$ when $|k| > N_0(a,b)$. We use an asymptotic expansion to show $W_k^R(b) < 0$. First, we find an expression for the asymptotic expansion of $\left|\hat{f}_b(k)\right|^2$. Using Lemma 3.5.2, we write

$$\begin{aligned} \left|\hat{f}_b(k)\right|^2 = \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{16\pi\sqrt{a^2+b^2}} &\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}}\right) \\ &\times \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b) - 2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b) + 2\pi a}{2\sqrt{a-\mathrm{i}b}}\right). \end{aligned}$$

Let z_1, z_2, z_3 and z_4 be given in (3.49). Then using Lemma 3.6.3, we write the asymptotic expansion of $\left|\hat{f}_b(k)\right|^2$ as

$$\begin{aligned} \left|\hat{f}_b(k)\right|^2 = \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{4\pi^2} &\left(\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b) - \mathrm{i}2\pi a)} - \frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b) + \mathrm{i}2\pi a)}\right. \\ &\left. + \frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b) + \mathrm{i}2\pi a)}\varepsilon_1(z_1) - \frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b) - \mathrm{i}2\pi a)}\varepsilon_1(z_2)\right) \\ \times &\left(\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b) + \mathrm{i}2\pi a)} - \frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b) - \mathrm{i}2\pi a)}\right. \\ &\left. + \frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b) - \mathrm{i}2\pi a)}\varepsilon_1(z_4) - \frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b) + \mathrm{i}2\pi a)}\varepsilon_1(z_3)\right). \end{aligned}$$

We write $\left|\hat{f}_b(k)\right|^2$ as a sum of approximation term and the remainder term

$$\left|\hat{f}_b(k)\right|^2 = \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{4\pi^2}\left(\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2\pi a)} - \frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\right) \times\left(\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)} - \frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)}\right) + \vartheta_k(b),$$

where

$$\begin{aligned}\vartheta_k(b) = \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{4\pi^2}\Bigg[&\left(\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2a\pi)} - \frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\right)\\ &\times\left(\frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_1(z_4) - \frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_1(z_3)\right)\\ &+\left(\frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_1(z_1) - \frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_1(z_2)\right)\\ &\times\left(\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)} - \frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)}\right)\\ &+\left(\frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_1(z_1) - \frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_1(z_2)\right)\\ &\times\left(\frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_1(z_4) - \frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_1(z_3)\right)\Bigg].\end{aligned} \tag{4.19}$$

Multiplying the expressions, we write $\left|\hat{f}_b(k)\right|^2$ as

$$\begin{aligned}\left|\hat{f}_b(k)\right|^2 = \frac{\mathrm{e}^{-2a\pi^2}}{4\pi^2}\Bigg(&\frac{1}{((k+2\pi b)-\mathrm{i}2\pi a)\,((k+2\pi b)+\mathrm{i}2\pi a)}\\ &-\frac{1}{((k+2\pi b)-\mathrm{i}2\pi a)\,((k-2\pi b)-\mathrm{i}2\pi a)}\\ &-\frac{1}{((k-2\pi b)+\mathrm{i}2\pi a)\,((k+2\pi b)+\mathrm{i}2\pi a)}\\ &+\frac{1}{((k-2\pi b)+\mathrm{i}2\pi a)\,((k-2\pi b)-\mathrm{i}2\pi a)}\Bigg) + \vartheta_k(b).\end{aligned}$$

Writing the above expression as a single fraction, we have

$$\left|\hat{f}_b(k)\right|^2 = \frac{16\pi^2(a^2+b^2)\mathrm{e}^{-2a\pi^2}}{((k+2\pi b)^2+4\pi^2a^2)\,((k-2\pi b)^2+4\pi^2a^2)} + \vartheta_k(b). \tag{4.20}$$

From (4.20), the asymptotic expansion of $\left|\hat{f}_0(k)\right|^2$ is calculated as

$$\left|\hat{f}_0(k)\right|^2 = \frac{16\pi^2 a^2 \mathrm{e}^{-2a\pi^2}}{\left(k^2+4\pi^2a^2\right)^2} + \vartheta_k(0). \tag{4.21}$$

Substituting (4.20) and (4.21) into (4.18) gives

$$\begin{aligned}
W_k^R(b) =& \frac{16\pi^2 a^2 \mathrm{e}^{-2a\pi^2}}{\left(k^2+4\pi^2a^2\right)^2} + \vartheta_k(0) \\
& - \frac{16\pi^2(a^2+b^2)\mathrm{e}^{-2a\pi^2}}{\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right)} - \vartheta_k(b) \\
=& \frac{-b^2\left(k^4+(4\pi a)^2k^2-(4\pi^2a)^2(a^2+b^2)\right)\mathrm{e}^{-2a\pi^2}}{\left(k^2+4\pi^2a^2\right)^2\left((k-2\pi b)^2+4\pi^2a^2\right)\left((k+2\pi b)^2+4\pi^2a^2\right)} \\
& + \vartheta_k(b) - \vartheta_k(0).
\end{aligned} \tag{4.22}$$

We write (4.22) as a sum of approximation term and the remainder term

$$W_k^R(b) = \widetilde{W}_k^R(b) + \vartheta_k(b) - \vartheta_k(0),$$

where the approximation term is given by

$$\widetilde{W}_k^R(b) = \frac{-b^2\left(k^4+(4\pi a)^2k^2-(4\pi^2a)^2(a^2+b^2)\right)\mathrm{e}^{-2a\pi^2}}{\left(k^2+4\pi^2a^2\right)^2\left((k-2\pi b)^2+4\pi^2a^2\right)\left((k+2\pi b)^2+4\pi^2a^2\right)}. \tag{4.23}$$

From (4.23), we see that $k^4+(4\pi a)^2k^2-(4\pi^2a)^2(a^2+b^2) > 0$ when $|k| > N_0(a,b)$. Therefore, $\widetilde{W}_k^R(b) < 0$ for all $a,\ b > 0$. To calculate the bounds for the remainder $\vartheta_k(b)-\vartheta_k(0)$, we find the bounds for $\vartheta_k(b)$ given in (4.19). Multiplication of expressions on the right side of (4.19) gives

$$\begin{aligned}
\vartheta_k(b) = \frac{\mathrm{e}^{-2a\pi^2}}{4\pi^2}\Bigg(& \frac{\varepsilon_1(z_4)}{\left((k+2\pi b)-\mathrm{i}2\pi a\right)\left((k-2\pi b)-\mathrm{i}2\pi a\right)} - \frac{\varepsilon_1(z_4)}{\left((k-2\pi b)^2+4\pi^2a^2\right)} \\
& - \frac{\varepsilon_1(z_3)}{\left((k+2\pi b)^2+4\pi^2a^2\right)} + \frac{\varepsilon_1(z_3)}{\left((k-2\pi b)+\mathrm{i}2\pi a\right)\left((k+2\pi b)+\mathrm{i}2\pi a\right)} \\
& + \frac{\varepsilon_1(z_1)}{\left((k-2\pi b)+\mathrm{i}2\pi a\right)\left((k+2\pi b)+\mathrm{i}2\pi a\right)} - \frac{\varepsilon_1(z_1)}{\left((k-2\pi b)^2+4\pi^2a^2\right)} \\
& - \frac{\varepsilon_1(z_2)}{\left((k+2\pi b)^2+4\pi^2a^2\right)} + \frac{\varepsilon_1(z_2)}{\left((k+2\pi b)-\mathrm{i}2\pi a\right)\left((k-2\pi b)-\mathrm{i}2\pi a\right)} \\
& + \frac{\varepsilon_1(z_1)\varepsilon_1(z_4)}{\left((k-2\pi b)^2+4\pi^2a^2\right)} - \frac{\varepsilon_1(z_1)\varepsilon_1(z_3)}{\left((k-2\pi b)+\mathrm{i}2\pi a\right)\left((k+2\pi b)+\mathrm{i}2\pi a\right)} \\
& - \frac{\varepsilon_1(z_2)\varepsilon_1(z_4)}{\left((k+2\pi b)-\mathrm{i}2\pi a\right)\left((k-2\pi b)-\mathrm{i}2\pi a\right)} + \frac{\varepsilon_1(z_2)\varepsilon_1(z_3)}{\left((k+2\pi b)^2+4\pi^2a^2\right)}\Bigg).
\end{aligned}$$

Arranging the like terms, we have

$$\begin{aligned}\vartheta_k(b) = \frac{\mathrm{e}^{-2a\pi^2}}{4\pi^2}\Bigg(& \frac{4\pi(\mathrm{i}a-b)\varepsilon_1(z_4)}{((k+2\pi b)-\mathrm{i}2\pi a)\,((k-2\pi b)^2+4\pi^2a^2)} \\ & + \frac{4\pi(-\mathrm{i}a+b)\varepsilon_1(z_3)}{((k-2\pi b)+\mathrm{i}2\pi a)\,((k+2\pi b)^2+4\pi^2a^2)} \\ & + \frac{4\pi(-\mathrm{i}a-b)\varepsilon_1(z_1)}{((k-2\pi b)^2+4\pi^2a^2)\,((k+2\pi b)+\mathrm{i}2\pi a)} \\ & + \frac{4\pi(\mathrm{i}a+b)\varepsilon_1(z_2)}{((k+2\pi b)^2+4\pi^2a^2)\,((k-2\pi b)-\mathrm{i}2\pi a)} \\ & - \frac{\varepsilon_1(z_1)\varepsilon_1(z_3)}{((k-2\pi b)+\mathrm{i}2\pi a)\,((k+2\pi b)+\mathrm{i}2\pi a)} \\ & - \frac{\varepsilon_1(z_2)\varepsilon_1(z_4)}{((k+2\pi b)-\mathrm{i}2\pi a)\,((k-2\pi b)-\mathrm{i}2\pi a)} \\ & + \frac{\varepsilon_1(z_1)\varepsilon_1(z_4)}{((k-2\pi b)^2+4\pi^2a^2)} + \frac{\varepsilon_1(z_2)\varepsilon_1(z_3)}{((k+2\pi b)^2+4\pi^2a^2)}\Bigg).\end{aligned}$$

Taking absolute value of the above expression

$$\begin{aligned}|\vartheta_k(b)| < \frac{\mathrm{e}^{-2a\pi^2}}{4\pi^2}\Bigg(& \frac{4\pi\sqrt{a^2+b^2}|\varepsilon_1(z_4)|}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\,((k-2\pi b)^2+4\pi^2a^2)} \\ & + \frac{4\pi\sqrt{a^2+b^2}|\varepsilon_1(z_3)|}{((k-2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\,((k+2\pi b)^2+4\pi^2a^2)} \\ & + \frac{4\pi\sqrt{a^2+b^2}|\varepsilon_1(z_1)|}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\,((k-2\pi b)^2+4\pi^2a^2)} \\ & + \frac{4\pi\sqrt{a^2+b^2}|\varepsilon_1(z_2)|}{((k-2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\,((k+2\pi b)^2+4\pi^2a^2)} \\ & + \frac{|\varepsilon_1(z_1)||\varepsilon_1(z_3)|}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\,((k-2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}} \\ & + \frac{|\varepsilon_1(z_2)||\varepsilon_1(z_4)|}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\,((k-2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}} \\ & + \frac{|\varepsilon_1(z_1)||\varepsilon_1(z_4)|}{((k-2\pi b)^2+4\pi^2a^2)} + \frac{|\varepsilon_1(z_2)||\varepsilon_1(z_3)|}{((k+2\pi b)^2+4\pi^2a^2)}\Bigg).\end{aligned}$$

From Lemmata 3.6.3 and 3.6.4, using the estimates of the remainder terms $\varepsilon_1(z_1)$, $\varepsilon_1(z_2)$,

$\varepsilon_1(z_3)$ and $\varepsilon_1(z_4)$, the absolute value $|\vartheta_k(b)|$ becomes

$$
\begin{aligned}
|\vartheta_k(b)| < \frac{\mathrm{e}^{-2a\pi^2}}{4\pi^2}\Bigg(& \frac{8\pi\sqrt{2}(a^2+b^2)}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right)^2} \\
&+\frac{8\pi\sqrt{2}(a^2+b^2)}{((k-2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right)^2} \\
&+\frac{8\pi(a^2+b^2)}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right)^2} \\
&+\frac{8\pi(a^2+b^2)}{((k-2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right)^2} \\
&+\frac{4\sqrt{2}(a^2+b^2)}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{3}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}} \\
&+\frac{4\sqrt{2}(a^2+b^2)}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{3}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}} \\
&+\frac{4\sqrt{2}(a^2+b^2)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^3}+\frac{4\sqrt{2}(a^2+b^2)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^3}\Bigg).
\end{aligned}
$$

Taking

$$
\frac{4(a^2+b^2)\mathrm{e}^{-2a\pi^2}}{\left((k+2\pi b)^2+4\pi^2a^2\right)^3\left((k-2\pi b)^2+4\pi^2a^2\right)^3}
$$

common from all expressions, we have

$$
\begin{aligned}
|\vartheta_k(b)| <& \frac{4(a^2+b^2)\mathrm{e}^{-2a\pi^2}}{\left((k+2\pi b)^2+4\pi^2a^2\right)^3\left((k-2\pi b)^2+4\pi^2a^2\right)^3} \\
&\times\Bigg(2\sqrt{2}\pi\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right) \\
&+2\sqrt{2}\pi\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right) \\
&+2\pi\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right) \\
&+2\pi\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right) \\
&+\sqrt{2}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}} \\
&+\sqrt{2}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}} \\
&+\sqrt{2}\left((k+2\pi b)^2+4\pi^2a^2\right)^3+\sqrt{2}\left((k-2\pi b)^2+4\pi^2a^2\right)^3\Bigg) \\
=&\mathcal{O}(k^{-5}).
\end{aligned}
$$

We estimate the remainder term in (4.22) as

$$\begin{aligned}|\vartheta_k(0) - \vartheta_k(b)| &< |\vartheta_k(0)| + |\vartheta_k(b)| \\ &= \mathcal{O}(k^{-5}) + \mathcal{O}(k^{-5}) = \mathcal{O}(k^{-5}).\end{aligned}$$

To prove that $\widetilde{W}_k^R(b)$ is an asymptotic expansion of $W_k^R(b)$, we write

$$\left|\frac{W_k^R(b)}{\widetilde{W}_k^R(b)} - 1\right| = \left|\frac{\theta_k(0) - \theta_k(b)}{\widetilde{W}_k^R(b)}\right| = \frac{|\theta_k(0) - \theta_k(b)|}{\left|\widetilde{W}_k^R(b)\right|} = \mathcal{O}\left(k^{-1}\right).$$

Therefore, $W_k^R(b) \sim \widetilde{W}_k^R(b)$ for all $a,\ b > 0$ when $|k| > N_0(a,b)$. The approximation term $\widetilde{W}_k^R(b) < 0$ and the remainder term has a faster decay than the approximation term. Hence, we conclude that $W_k^R(b) < 0$ for all $a,\ b > 0$ when $|k| > N_0(a,b)$.

For $\alpha < \alpha^*$

Let $E_k^L(\alpha)$ be a function given by

$$E_k^L(\alpha) = \left|\hat{f}_\alpha(k)\right|^2 - \left|\hat{f}_{\alpha^*}(k)\right|^2 = -\left(\left|\hat{f}_{\alpha^*}(k)\right|^2 - \left|\hat{f}_\alpha(k)\right|^2\right). \tag{4.24}$$

Substituting (4.15) and (4.16) into (4.24), we have

$$\begin{aligned}E_k^L(\alpha) = -\Bigg(&\frac{\mathrm{e}^{\frac{-k^2}{2\pi\gamma}}}{16\pi^2\gamma}\left|\operatorname{erf}\left(\frac{\mathrm{i}k - 2\pi^2\gamma}{2\sqrt{\pi\gamma}}, \frac{\mathrm{i}k + 2\pi^2\gamma}{2\sqrt{\pi\gamma}}\right)\right|^2 - \frac{\mathrm{e}^{\frac{-\pi\gamma k^2}{2\left(\pi^2\gamma^2 + \frac{1}{4}(2\pi\mu - \cot\alpha)^2\right)}}}{16\pi\sqrt{\pi^2\gamma^2 + \frac{1}{4}(2\pi\mu - \cot\alpha)^2}} \\ &\times \left|\operatorname{erf}\left(\frac{\mathrm{i}\,(k - \pi(2\pi\mu - \cot\alpha)) - 2\pi^2\gamma}{2\sqrt{\pi\gamma + \frac{\mathrm{i}}{2}(2\pi\mu - \cot\alpha)}}, \frac{\mathrm{i}\,(k + \pi(2\pi\mu - \cot\alpha)) + 2\pi^2\gamma}{2\sqrt{\pi\gamma + \frac{\mathrm{i}}{2}(2\pi\mu - \cot\alpha)}}\right)\right|^2\Bigg).\end{aligned}$$

For given $\gamma > 0$ and $\mu \in \mathbb{R}^+$, we shall show that $E_k^L(\alpha) > 0$ when $|k| > N(\gamma,\mu)$. We simplify the above expression using the substitution (4.17). When $0 < \alpha < \alpha^*$, then $a > 0$ and $b < 0$, and $E_k^L(\alpha)$ becomes $W_k^L(b)$, given by

$$\begin{aligned}W_k^L(b) = -\Bigg(&\frac{\mathrm{e}^{\frac{-k^2}{2a}}}{16\pi a}\left|\operatorname{erf}\left(\frac{\mathrm{i}k - 2\pi a}{2\sqrt{a}}, \frac{\mathrm{i}k + 2\pi a}{2\sqrt{a}}\right)\right|^2 \\ &- \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{16\pi\sqrt{a^2+b^2}}\left|\operatorname{erf}\left(\frac{\mathrm{i}(k - 2\pi b) - 2\pi a}{2\sqrt{a + \mathrm{i}b}}, \frac{\mathrm{i}(k + 2\pi b) + 2\pi a}{2\sqrt{a + \mathrm{i}b}}\right)\right|^2\Bigg).\end{aligned} \tag{4.25}$$

We equivalently prove that $W_k^L(b) > 0$ for all $a > 0$ and $b < 0$ when $|k| > N_0(a,b)$. Similar calculations as in the case $\alpha > \alpha^*$ are performed in (4.25) which gives

$$\begin{aligned}W_k^L(b) =& \frac{b^2\left(k^4 + (4\pi a)^2 k^2 - (4\pi^2 a)^2(a^2 + b^2)\right)\mathrm{e}^{-2a\pi^2}}{\left(k^2 + 4\pi^2 a^2\right)^2\left((k - 2\pi b)^2 + 4\pi^2 a^2\right)\left((k + 2\pi b)^2 + 4\pi^2 a^2\right)} \\ &+ \vartheta_k(0) - \vartheta_k(b).\end{aligned} \tag{4.26}$$

We write (4.26) as a sum of approximation term and the remainder term

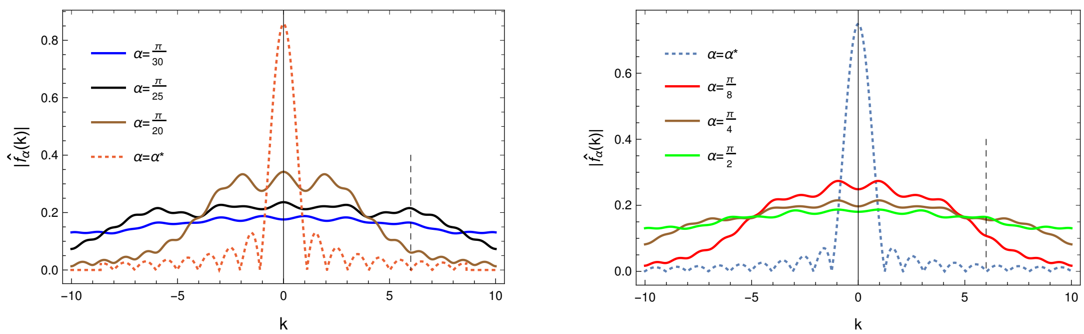
$$W_k^L(b) = \widetilde{W}_k^L(b) + \vartheta_k(0) - \vartheta_k(b),$$

where the approximation term is given by

$$\widetilde{W}_k^L(b) = \frac{b^2\left(k^4 + (4\pi a)^2k^2 - (4\pi^2 a)^2(a^2+b^2)\right)\mathrm{e}^{-2a\pi^2}}{\left(k^2+4\pi^2a^2\right)^2\left((k-2\pi b)^2+4\pi^2a^2\right)\left((k+2\pi b)^2+4\pi^2a^2\right)}. \tag{4.27}$$

From (4.27), we see that $k^4 + (4\pi a)^2k^2 - (4\pi^2 a)^2(a^2+b^2) > 0$ when $|k| > N_0(a,b)$. Therefore, $\widetilde{W}_k^L(b) > 0$ for all $a > 0$ and $b < 0$. Performing the similar calculations as in case $\alpha > \alpha^*$, we can show $W_k^L(b) \sim \widetilde{W}_k^L(b)$ for all $a > 0$ and $b < 0$ when $|k| > N_0(a,b)$. Hence, $W_k^L(b) > 0$ for all $a > 0$ and $b < 0$. ■

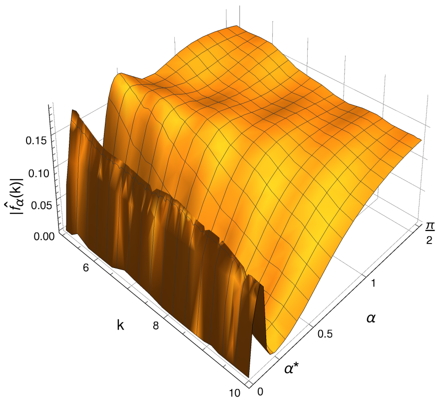
In Theorem 4.2.2, we have proved that for given $\gamma > 0$ and $\mu \in \mathbb{R}^+$, the absolute value of the fractional Fourier coefficients $\left|\hat{f}_{\alpha^*}(k)\right|$ for $|k| > N(\gamma,\mu)$ is the minimum. The absolute value of the fractional Fourier coefficients $\left|\hat{f}_\alpha(k)\right|$ for various fractional Fourier parameters is illustrated in Figure 4.2.



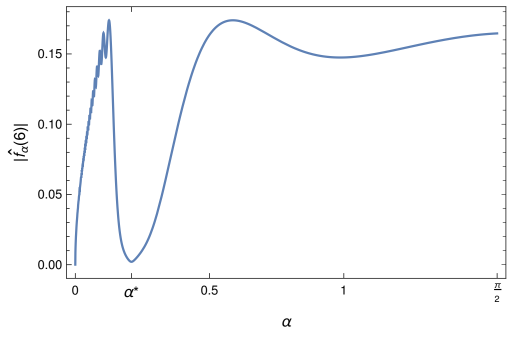
(a) (b)

Figure 4.2: For the chirp $f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}$ with $\gamma = \frac{1}{10\pi}$ and $\mu = \frac{3}{4}$, the graph of $\left|\hat{f}_\alpha(k)\right|$ over $-10 \le k \le 10$ (a) for some α on the domain $0 < \alpha \le \alpha^* \approx 0.209$, and (b) for some α on the domain $0.209 \approx \alpha^* \le \alpha \le \frac{\pi}{2}$.

The three-dimensional behavior of $\left|\hat{f}_\alpha(k)\right|$ for $5 \le k \le 10$ on $0 < \alpha \le \frac{\pi}{2}$ and plot of $\left|\hat{f}_\alpha(6)\right|$ on $0 < \alpha \le \frac{\pi}{2}$ is shown in Figure 4.3. We observe that when the parameter α lies close to zero, the fractional Fourier coefficients $\left|\hat{f}_\alpha(k)\right|$ for $|k| > N(\gamma,\mu)$ become oscillatory with respect to α.



(a)



(b)

Figure 4.3: For the chirp $f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}$ with $\gamma = \frac{1}{10\pi}$ and $\mu = \frac{3}{4}$, (a) the continuum of $\left|\hat{f}_\alpha(k)\right|$ for $5 \leq k \leq 10$ on fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$, and (b) the graph of $\left|\hat{f}_\alpha(6)\right|$ on $0 < \alpha \leq \frac{\pi}{2}$.

4.2.3 The Minimality in L^2 Error by FrFS Approximation

The approximation of exponential linear chirps by the partial sums of FrFS with optimal fractional Fourier parameter gives the minimum L^2 error for sufficiently large $N \in \mathbb{N}_0$, which we prove in this section. We describe the problem of the best approximation in the following way: We are given an exponential linear chirp f_γ^μ, the complex exponentials $\phi_{k,\alpha}$ and the L^2 norm $\|\cdot\|_{L^2[-\pi,\pi]}$. We take all possible basis $\{\phi_{k,\alpha} : k \in \mathbb{Z}\}$ with $0 < \alpha \leq \frac{\pi}{2}$ and form the corresponding partial sums $S_{N,\alpha}$, where

$$S_{N,\alpha}(t) = \sum_{|k|\leq N} \hat{f}_\alpha(k)\phi_{k,\alpha}(t), \quad N \in \mathbb{N}_0.$$

There is a parameter α^* in the fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$ so that the partial sum S_{n,α^*} is the closest fit to f_γ^μ on $[-\pi, \pi]$ for large $N \in \mathbb{N}_0$.

Corollary. *Let*

$$f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ and $\mu \in \mathbb{R}^+$. There exists $N \in \mathbb{N}_0$ such that for $N > N(\gamma, \mu)$, the optimal fractional parameter α^* provides the minimum L^2 error on $[-\pi, \pi]$. i.e.,

$$\min_{0<\alpha\leq\frac{\pi}{2}} \left\|f_\gamma^\mu - S_{N,\alpha}\right\|_{L^2[-\pi,\pi]} = \left\|f_\gamma^\mu - S_{N,\alpha^*}\right\|_{L^2[-\pi,\pi]}.$$

Proof. We divide the fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$ into two parts namely $\alpha > \alpha^*$ and $\alpha < \alpha^*$, and prove

$$\left\|f_\gamma^\mu - S_{N,\alpha^*}\right\|_{L^2[-\pi,\pi]} < \left\|f_\gamma^\mu - S_{N,\alpha}\right\|_{L^2[-\pi,\pi]}$$

on $0 < \alpha \leq \frac{\pi}{2}$ when $\alpha \neq \alpha^*$.

For $\alpha > \alpha^*$

Let $J_N^R(\alpha)$ be a function given by

$$J_N^R(\alpha) = \left\|f_\gamma^\mu - S_{N,\alpha^*}\right\|^2_{L^2[-\pi,\pi]} - \left\|f_\gamma^\mu - S_{N,\alpha}\right\|^2_{L^2[-\pi,\pi]}.$$

Using Parseval's identity, $J_N^R(\alpha)$ becomes

$$J_N^R(\alpha) = \sum_{|k|>N}\left|\hat{f}_{\alpha^*}(k)\right|^2 - \sum_{|k|>N}\left|\hat{f}_{\alpha}(k)\right|^2 = \sum_{|k|>N}\left(\left|\hat{f}_{\alpha^*}(k)\right|^2 - \left|\hat{f}_{\alpha}(k)\right|^2\right).$$

Using Theorem 4.2.2, we conclude $J_N^R(\alpha) < 0$ when $N > N(\gamma,\mu)$ on $\alpha^* < \alpha \leq \frac{\pi}{2}$.

For $\alpha < \alpha^*$

Let $J_N^L(\alpha)$ be a function given by

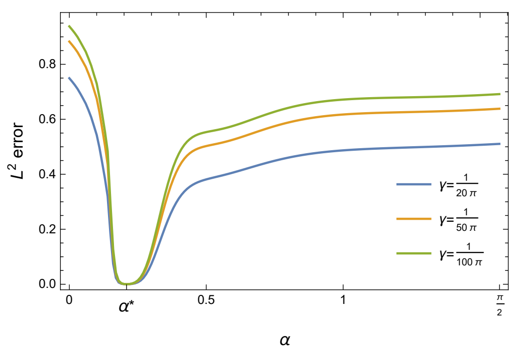
$$J_N^L(\alpha) = \left\|f_\gamma^\mu - S_{N,\alpha}\right\|^2_{L^2[-\pi,\pi]} - \left\|f_\gamma^\mu - S_{N,\alpha^*}\right\|^2_{L^2[-\pi,\pi]}.$$

Using Parseval's identity, $J_N^L(\alpha)$ becomes

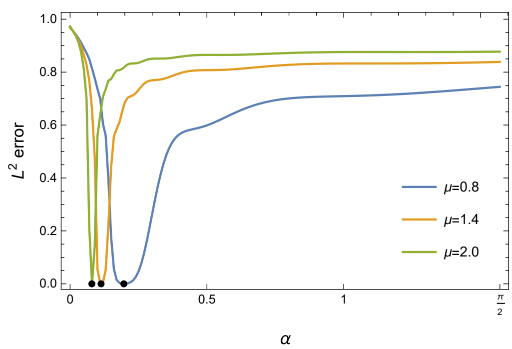
$$J_N^L(\alpha) = \sum_{|k|>N}\left|\hat{f}_{\alpha}(k)\right|^2 - \sum_{|k|>N}\left|\hat{f}_{\alpha^*}(k)\right|^2 = \sum_{|k|>N}\left(\left|\hat{f}_{\alpha}(k)\right|^2 - \left|\hat{f}_{\alpha^*}(k)\right|^2\right).$$

Using Theorem 4.2.2, we conclude $J_N^L(\alpha) > 0$ when $N > N(\gamma,\mu)$ on $0 < \alpha < \alpha^*$. ■

Hence, the partial sum S_{N,α^*} provides the best approximation to f_γ^μ in the fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$ relative to the norm $\|\cdot\|_{L^2[-\pi,\pi]}$ for large $N \in \mathbb{N}_0$. The plot of L^2 error is shown in Figure 4.4.



(a)



(b)

Figure 4.4: For the chirp $f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}$, the L^2 error in FrFS approximation of exponential linear chirps by using $N = 3$ with (a) $\mu = \frac{3}{4}$, and (b) $\gamma = \frac{1}{200\pi}$.

4.3 Monotonicity Properties of Fractional Fourier Coefficients

This section is devoted to the monotonicity properties of fractional Fourier coefficients of exponential linear chirps. We define a domain $\mathcal{D}$ by

$$\mathcal{D} = \left\{(\gamma,\mu) : \gamma \geq \frac{1}{2\pi},\ 0 < \mu \leq \frac{\sqrt{\gamma}}{8\pi}\left(\lambda \mathrm{e}^{\pi^3\gamma} + \sqrt{\left(\lambda \mathrm{e}^{\pi^3\gamma}\right)^2 - 64\pi^2\gamma}\right)\right\}, \tag{4.28}$$

where

$$\lambda = \left(1 - \frac{2\sqrt{2}\mathrm{e}^{-\frac{\pi^2}{2}}}{\pi^{\frac{3}{2}}}\right)^2.$$

The fractional Fourier coefficients of zero degree, the fractional Fourier coefficients of large degree and the L^2 error are monotone with respect to α. The sense of monotonicity differs whether $\alpha > \alpha^*$ or $\alpha < \alpha^*$.

4.3.1 Fractional Fourier Coefficients of Zero Degree

The absolute value of the fractional Fourier coefficients $\left|\hat{f}_\alpha(0)\right|$ is monotone on the domain $\mathcal{D}$. For $\alpha > \alpha^*$, fractional Fourier coefficients $\left|\hat{f}_\alpha(0)\right|$ are monotonic decreasing whereas the coefficients are monotonic increasing for $\alpha_1 \leq \alpha < \alpha^*$, where $0 < \alpha_1 < \alpha^*$. Now we prove the monotonicity of $\left|\hat{f}_\alpha(0)\right|$ in the following theorem.

Theorem 4.3.1. *Let*

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi,\pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ *and* $\mu \in \mathbb{R}^+$. *Let* $(\gamma,\mu) \in \mathcal{D}$ *given in* (4.28). *Then there exists* $\alpha_1 = \alpha_1(\gamma,\mu)$ *such that*

$$\left|\hat{f}_{\alpha_R}(0)\right| > \left|\hat{f}_{\beta_R}(0)\right| \quad \text{for} \quad \alpha^* < \alpha_R < \beta_R \leq \frac{\pi}{2}$$

and

$$\left|\hat{f}_{\beta_L}(0)\right| < \left|\hat{f}_{\alpha_L}(0)\right| \quad \text{for} \quad \alpha_1 \leq \beta_L < \alpha_L < \alpha^*,$$

where $0 < \alpha_1 < \alpha^*$ *and* α^* *is the optimal fractional Fourier parameter.*

Proof. Let $\left|\hat{f}_\alpha(0)\right|^2$ be denoted by $H(\alpha)$. From (4.4), we write

$$H(\alpha) = \frac{1}{4\pi\sqrt{\pi^2\gamma^2 + \frac{1}{4}(2\pi\mu - \cot\alpha)^2}}\left|\mathrm{erf}\left(\pi\sqrt{\pi\gamma + \frac{\mathrm{i}}{2}(2\pi\mu - \cot\alpha)}\right)\right|^2. \tag{4.29}$$

The necessary and sufficient condition of $H(\alpha)$ to be monotone for all $(\gamma, \mu) \in \mathcal{D}$ is

$$\begin{aligned} H'(\alpha) &< 0 \quad \text{on} \quad \alpha^* < \alpha \leq \frac{\pi}{2}, \\ H'(\alpha) &> 0 \quad \text{on} \quad \alpha_1 \leq \alpha < \alpha^*, \end{aligned} \tag{4.30}$$

where we choose

$$\alpha_1 = \arctan\left(\frac{1}{2\pi\mu + \frac{\sqrt{\gamma}}{8}\left(\lambda e^{\pi^3\gamma} + \sqrt{\left(\lambda e^{a\pi^3\gamma}\right)^2 - 64\pi^2\gamma}\right)}\right).$$

We simplify (4.29) using the substitution

$$\pi\gamma = a(\gamma) \quad \text{and} \quad \frac{1}{2}(2\pi\mu - \cot\alpha) = b(\alpha). \tag{4.31}$$

For the sake of convenience, we write a and b instead of $a(\gamma)$ and $b(\alpha)$. When $\alpha^* < \alpha \leq \frac{\pi}{2}$, then $0 < b \leq \pi\mu$. The domain $\mathcal{D}$ in (4.28) becomes Ω_R as defined in (3.15)

$$\Omega_R = \left\{(a, b) : a \geq \frac{1}{2},\ 0 < b \leq \frac{\sqrt{a\pi}}{8\pi}\left(\lambda e^{a\pi^2} + \sqrt{\left(\lambda e^{a\pi^2}\right)^2 - 64a\pi}\right)\right\},$$

where λ is given in (3.16). When $\alpha_1 \leq \alpha < \alpha^*$, the domain $\mathcal{D}$ in (4.28) becomes Ω_L, where

$$\Omega_L = \left\{(a, b) : a \geq \frac{1}{2},\ \frac{-\sqrt{a\pi}}{8\pi}\left(\lambda e^{a\pi^2} + \sqrt{\left(\lambda e^{a\pi^2}\right)^2 - 64a\pi}\right) \leq b < 0\right\}.$$

We calculate the range of domain Ω_L as follows. The domain $\alpha_1 \leq \alpha < \alpha^*$ is written as

$$\arctan\left(\frac{1}{2\pi\mu + \frac{\sqrt{\gamma}}{8}\left(\lambda e^{\pi^3\gamma} + \sqrt{\left(\lambda e^{a\pi^3\gamma}\right)^2 - 64\pi^2\gamma}\right)}\right) \leq \alpha < \arctan\left(\frac{1}{2\pi\mu}\right),$$

which is equivalent to

$$\frac{1}{2\pi\mu + \frac{\sqrt{\gamma}}{8}\left(\lambda e^{\pi^3\gamma} + \sqrt{\left(\lambda e^{a\pi^3\gamma}\right)^2 - 64\pi^2\gamma}\right)} \leq \tan\alpha < \frac{1}{2\pi\mu}.$$

The range of $\cot\alpha$ is

$$2\pi\mu < \cot\alpha \leq 2\pi\mu + \frac{\sqrt{\gamma}}{8}\left(\lambda e^{\pi^3\gamma} + \sqrt{\left(\lambda e^{a\pi^3\gamma}\right)^2 - 64\pi^2\gamma}\right).$$

The range of $\frac{1}{2}(2\pi\mu - \cot\alpha_1)$ is

$$\frac{-\sqrt{\gamma}}{8}\left(\lambda e^{\pi^3\gamma} + \sqrt{\left(\lambda e^{a\pi^3\gamma}\right)^2 - 64\pi^2\gamma}\right) \leq \frac{1}{2}(2\pi\mu - \cot\alpha) < 0.$$

Using the substitution (4.31), we have

$$\frac{-\sqrt{a\pi}}{8\pi}\left(\lambda e^{a\pi^2} + \sqrt{\left(\lambda e^{a\pi^2}\right)^2 - 64a\pi}\right) \leq b < 0.$$

The necessary and sufficient condition (4.30) is equivalent to

$$\begin{aligned} M'(b) < 0 \quad &\text{for all} \quad (a,b) \in \Omega_R, \\ M'(b) > 0 \quad &\text{for all} \quad (a,b) \in \Omega_L. \end{aligned}$$

The function $H(\alpha)$ in (4.29) becomes $M(b)$, where

$$\begin{aligned} M(b) &= \frac{1}{4\pi\sqrt{a^2+b^2}}\left|\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\right|^2 \\ &= \frac{1}{4\pi\sqrt{a^2+b^2}}\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right). \end{aligned} \tag{4.32}$$

The domain $\boldsymbol{\Omega_R}$

We shall show that $M'(b) < 0$ for all $(a,b) \in \Omega_R$. From (4.32), we have

$$\begin{aligned} M'(b) =& \frac{1}{4\pi\sqrt{a^2+b^2}}\left(\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\frac{2}{\sqrt{\pi}}e^{-\pi^2(a-\mathrm{i}b)}\frac{(-\mathrm{i})\pi}{2\sqrt{a-\mathrm{i}b}}\right. \\ &\left. + \operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\frac{2}{\sqrt{\pi}}e^{-\pi^2(a+\mathrm{i}b)}\frac{\mathrm{i}\pi}{2\sqrt{a+\mathrm{i}b}}\right) \\ &+ \operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\left(\frac{-2b}{8\pi\left(a^2+b^2\right)^{\frac{3}{2}}}\right). \end{aligned}$$

We arrange the terms in the form of $\frac{\operatorname{erf}(z)}{z}$ and simplify $M'(b)$ to get

$$\begin{aligned} M'(b) =& \frac{1}{4\pi\sqrt{a^2+b^2}}\left(\frac{\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)}{\pi\sqrt{a+\mathrm{i}b}}e^{-\pi^2(a-\mathrm{i}b)}\frac{(-\mathrm{i})\pi\sqrt{\pi}(a+\mathrm{i}b)}{\sqrt{a^2+b^2}}\right. \\ &\left. + \frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}}e^{-\pi^2(a+\mathrm{i}b)}\frac{\mathrm{i}\pi\sqrt{\pi}(a-\mathrm{i}b)}{\sqrt{a^2+b^2}}\right) \\ &+ \frac{\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)}{\pi\sqrt{a+\mathrm{i}b}}\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}}\left(\frac{-b\pi}{4(a^2+b^2)}\right). \end{aligned}$$

Multiplying and divide the right side of $M'(b)$ by $\frac{-1}{4(a^2+b^2)}$, we get

$$M'(b) = \frac{-1}{4(a^2+b^2)}\left(b\pi\left|\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}}\right|^2 + \mathrm{i}\sqrt{\pi}(a+\mathrm{i}b)\frac{\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)}{\pi\sqrt{a+\mathrm{i}b}}\mathrm{e}^{-\pi^2(a-\mathrm{i}b)} - \mathrm{i}\sqrt{\pi}(a-\mathrm{i}b)\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}}\mathrm{e}^{-\pi^2(a+\mathrm{i}b)}\right). \tag{4.33}$$

From Lemma 3.3.1, we have

$$\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}} = I_c + \mathrm{i}I_s. \tag{4.34}$$

Moreover, we write the expression for the absolute value of (4.34)

$$\left|\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}}\right|^2 = \frac{\left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2}{\pi^2\sqrt{a^2+b^2}} = I_c^2 + I_s^2, \tag{4.35}$$

where I_c and I_s are integrals of the form

$$I_c = \frac{2}{\sqrt{\pi}}\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,\mathrm{d}u > 0 \tag{4.36}$$

and

$$I_s = \frac{2}{\sqrt{\pi}}\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u > 0 \tag{4.37}$$

respectively. Using (4.34) and (4.35), (4.33) simplifies to

$$\begin{aligned} M'(b) &= \frac{-1}{4(a^2+b^2)}\Big(b\pi\left(I_c^2+I_s^2\right) + \mathrm{i}\sqrt{\pi}\,(a+\mathrm{i}b)\,(I_c-\mathrm{i}I_s)\,\mathrm{e}^{-\pi^2(a-\mathrm{i}b)} \\ &\qquad - \mathrm{i}\sqrt{\pi}\,(a-\mathrm{i}b)\,(I_c+\mathrm{i}I_s)\,\mathrm{e}^{-\pi^2(a+\mathrm{i}b)}\Big) \\ &= \frac{-1}{4(a^2+b^2)}\Big(b\pi\left(I_c^2+I_s^2\right) + \sqrt{\pi}\mathrm{e}^{-a\pi^2}\,(a+\mathrm{i}b)\,(\mathrm{i}I_c+I_s)\,\mathrm{e}^{\mathrm{i}b\pi^2} \\ &\qquad - \sqrt{\pi}\mathrm{e}^{-a\pi^2}\,(a-\mathrm{i}b)\,(\mathrm{i}I_c-I_s)\,\mathrm{e}^{-\mathrm{i}b\pi^2}\Big). \end{aligned}$$

Using Euler's identity $\mathrm{e}^{ib\pi^2} = \cos b\pi^2 + \mathrm{i}\sin b\pi^2$, the derivative $M'(b)$ becomes

$$\begin{aligned} M'(b) &= \frac{-1}{4(a^2+b^2)}\Big(b\pi\left(I_c^2+I_s^2\right) + \sqrt{\pi}\mathrm{e}^{-a\pi^2}\,(a+\mathrm{i}b)\left(\cos b\pi^2+\mathrm{i}\sin b\pi^2\right)(\mathrm{i}I_c+I_s) \\ &\qquad - \sqrt{\pi}\mathrm{e}^{-a\pi^2}\,(a-\mathrm{i}b)\left(\cos b\pi^2-\mathrm{i}\sin b\pi^2\right)(\mathrm{i}I_c-I_s)\Big) \\ &= \frac{-1}{4(a^2+b^2)}\Big(b\pi\left(I_c^2+I_s^2\right) - 2\sqrt{\pi}\mathrm{e}^{-a\pi^2}\left(a\sin b\pi^2+b\cos b\pi^2\right)I_c \\ &\qquad - 2\sqrt{\pi}\mathrm{e}^{-a\pi^2}\left(b\sin b\pi^2-a\cos b\pi^2\right)I_s\Big). \end{aligned} \tag{4.38}$$

Since $b \neq 0$, multiplying and dividing right side of (4.38) by b gives

$$M'(b) = \frac{-b}{4(a^2+b^2)}\Bigg(\pi\left(I_c^2+I_s^2\right) - \frac{2\sqrt{\pi}\mathrm{e}^{-a\pi^2}}{b}\left(a\sin b\pi^2 + b\cos b\pi^2\right)I_c - \frac{2\sqrt{\pi}\mathrm{e}^{-a\pi^2}}{b}\left(b\sin b\pi^2 - a\cos b\pi^2\right)I_s\Bigg). \tag{4.39}$$

Using (4.35), (4.36) and (4.37), we write (4.39) in terms of absolute value of the error function and integrals

$$\begin{aligned} M'(b) =& \frac{-b}{4(a^2+b^2)}\Bigg(\pi\frac{\left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2}{\pi^2\sqrt{a^2+b^2}} \\ &- \frac{4\mathrm{e}^{-a\pi^2}}{b}\left(a\sin b\pi^2 + b\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,\mathrm{d}u \\ &- \frac{4\mathrm{e}^{-a\pi^2}}{b}\left(b\sin b\pi^2 - a\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u\Bigg) \\ =& \frac{-b}{8\pi(a^2+b^2)^{\frac{3}{2}}}\Bigg(2\left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 \\ &- \frac{8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}}{b}\left(a\sin b\pi^2 + b\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,\mathrm{d}u \\ &- \frac{8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}}{b}\left(b\sin b\pi^2 - a\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u\Bigg). \end{aligned} \tag{4.40}$$

We write (4.40) into the following form

$$M'(b) = \frac{-b}{8\pi(a^2+b^2)^{\frac{3}{2}}}\left(\left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 - \xi_1(a,b) + \left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 - \xi_2(a,b)\right), \tag{4.41}$$

where

$$\xi_1(a,b) = \frac{8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}}{b}\left(a\sin b\pi^2 + b\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\cos\left(b\pi^2u^2\right)\,\mathrm{d}u$$

and

$$\xi_2(a,b) = \frac{8\pi\sqrt{a^2+b^2}\mathrm{e}^{-a\pi^2}}{b}\left(b\sin b\pi^2 - a\cos b\pi^2\right)\int_0^1 \mathrm{e}^{-a\pi^2u^2}\sin\left(b\pi^2u^2\right)\,\mathrm{d}u.$$

From (4.41), the derivative $M'(b) < 0$ if both inequalities

$$\left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 > \xi_1(a,b) \tag{4.42}$$

and

$$\left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 > \xi_2(a,b) \tag{4.43}$$

are satisfied. We prove that both inequalities (4.42) and (4.43) hold for $(a,b) \in \Omega_R$. From Lemmata 3.4.1 and 3.4.2, we write

$$\inf_{(a,b)\in\Omega_R} \left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 > \sup_{(a,b)\in\Omega_R} \xi_1(a,b). \tag{4.44}$$

Since the inequality (4.44) holds for all $(a,b) \in \Omega_R$, we have

$$\left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 > \xi_1(a,b).$$

Therefore, the inequality (4.42) holds for Ω_R. Again from Lemmata 3.4.1 and 3.4.2, we write

$$\inf_{(a,b)\in\Omega_R} \left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 > \sup_{(a,b)\in\Omega_R} \xi_2(a,b). \tag{4.45}$$

Since the inequality (4.45) holds for all $(a,b) \in \Omega_R$, we have

$$\left|\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)\right|^2 > \xi_2(a,b).$$

Therefore, the inequality (4.43) holds for Ω_R. Since both inequalities (4.42) and (4.43) hold for all $(a,b) \in \Omega_R$, we conclude $M'(b) < 0$ for all $(a,b) \in \Omega_R$.

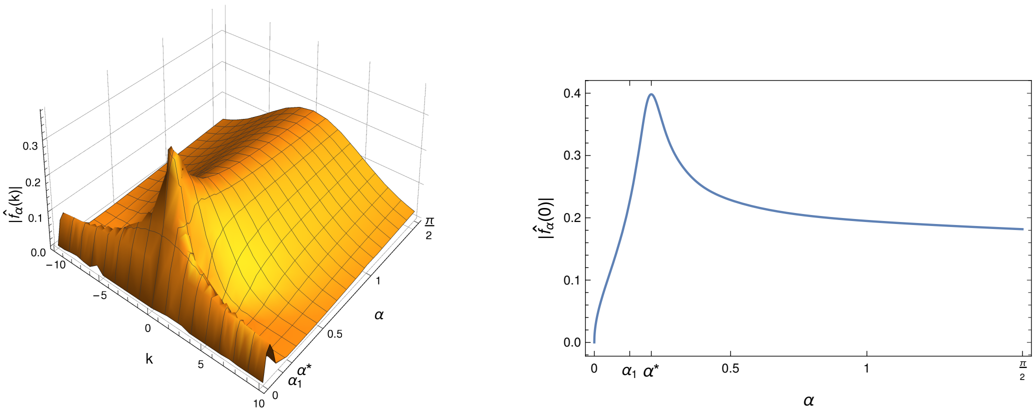
The domain Ω_L

Now we prove that $M'(b) > 0$ for all $(a,b) \in \Omega_L$. We find $M'(-b)$ by replacing b by $-b$ in (4.33)

$$\begin{aligned}
M'(-b) =& \frac{-1}{4(a^2+b^2)}\Bigg(-b\pi\frac{\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)}{\pi\sqrt{a+\mathrm{i}b}}\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}} \\
&+ \mathrm{i}\sqrt{\pi}(a-\mathrm{i}b)\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}}\mathrm{e}^{-\pi^2(a+\mathrm{i}b)} - \mathrm{i}\sqrt{\pi}(a+\mathrm{i}b)\frac{\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)}{\pi\sqrt{a+\mathrm{i}b}}\mathrm{e}^{-\pi^2(a-\mathrm{i}b)}\Bigg) \\
=& \frac{1}{4(a^2+b^2)}\Bigg(b\pi\frac{\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)}{\pi\sqrt{a+\mathrm{i}b}}\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}} \\
&+ \mathrm{i}\sqrt{\pi}(a+\mathrm{i}b)\frac{\operatorname{erf}\left(\pi\sqrt{a+\mathrm{i}b}\right)}{\pi\sqrt{a+\mathrm{i}b}}\mathrm{e}^{-\pi^2(a-\mathrm{i}b)} - \mathrm{i}\sqrt{\pi}(a-\mathrm{i}b)\frac{\operatorname{erf}\left(\pi\sqrt{a-\mathrm{i}b}\right)}{\pi\sqrt{a-\mathrm{i}b}}\mathrm{e}^{-\pi^2(a+\mathrm{i}b)}\Bigg) \\
=& -M'(b).
\end{aligned}$$

Since $M'(-b) = -M'(b)$ and we have already proved that $M'(b) < 0$ for all $(a,b) \in \Omega_R$, therefore $M'(-b) > 0$ for all $(a,b) \in \Omega_L$. ∎

We have proved that the fractional Fourier coefficients $\left|\hat{f}_\alpha(0)\right|$ have monotonic behavior on the domain $\mathcal{D}$. The monotone behavior of fractional Fourier coefficients is illustrated in Figure 4.5.



(a) (b)

Figure 4.5: For the chirp $f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}$ with $\gamma = \frac{1}{2\pi}$, (a) the three-dimensional plot of $\left|\hat{f}_\alpha(k)\right|$ for $-10 \le k \le 10$ on fractional Fourier parameter domain $0 < \alpha \le \frac{\pi}{2}$ and, (b) the graph of $\left|\hat{f}_\alpha(0)\right|$ on $0 < \alpha \le \frac{\pi}{2}$.

Numerical Validity of Theorem 4.3.1

In Theorem 4.3.1, we have proved the monotonicity on the domain Ω given by

$$\Omega = \left\{(a,b) : a \ge \frac{1}{2},\ -\Psi(a) \le b \le \Psi(a)\right\} = \Omega_L \cup \Omega_R,$$

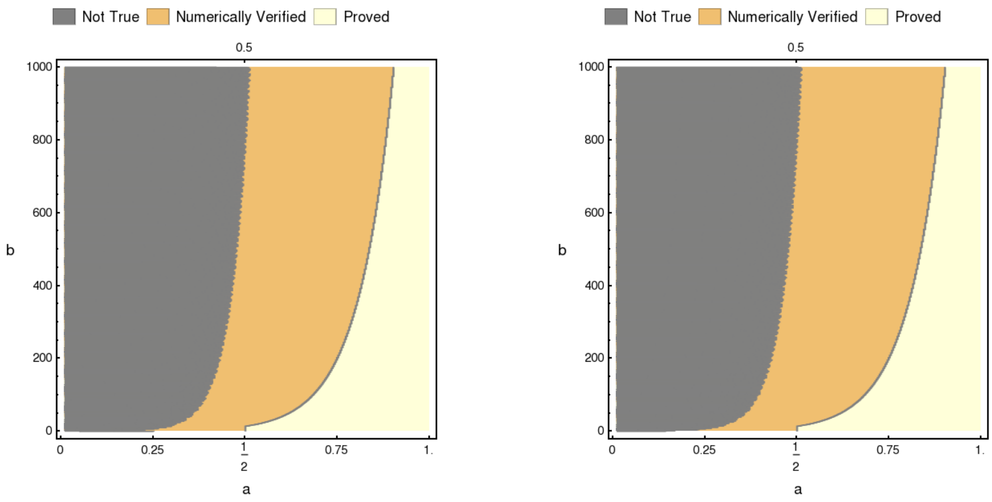
where

$$\Psi(a) = \frac{\sqrt{a\pi}}{8\pi}\left(\lambda \mathrm{e}^{a\pi^2} + \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right) \quad \text{and} \quad \lambda = \left(1 - \frac{2\sqrt{2}\mathrm{e}^{\frac{-\pi^2}{2}}}{\pi^{\frac{3}{2}}}\right)^2.$$

We have performed the numerical experiments which reveal that Theorem 4.3.1 is valid on a larger domain ∂ given by

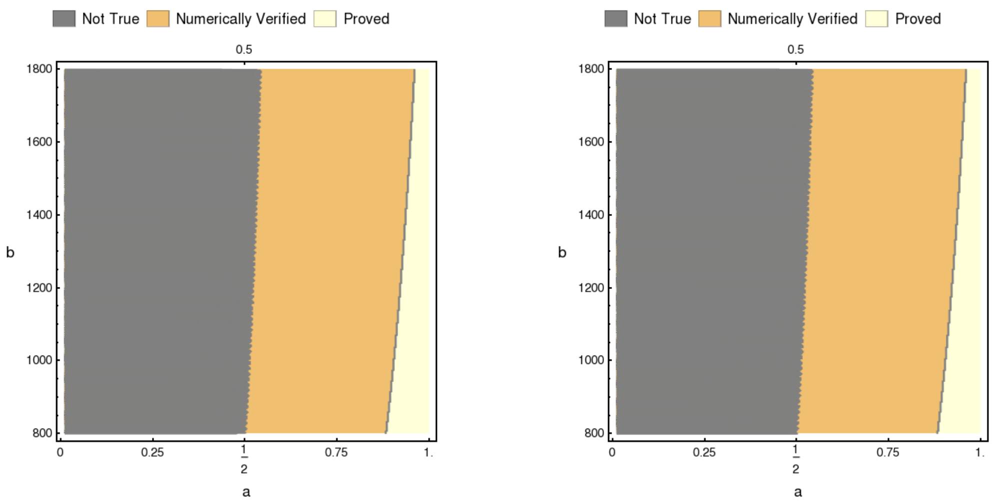
$$\partial = \left\{(a,b) : a \ge \frac{1}{2},\ b_L(a) \le b \le b_R(a)\right\}.$$

We confirm the numerical validity of the monotonicity using ListContourPlot in Wolfram Mathematica 11.3 by plotting inequalities (3.20) and (3.21) over the domain (3.15). In Figures 4.6, 4.7 and 4.8, the rightmost areas indicate the region where the inequalities (3.20) and (3.21) are proved. The leftmost areas indicate the region where the inequalities (3.20) and (3.21) are not true. The areas in the center of figures is the region where monotonicty is numerically verified. In Figure 4.8, we observe the oscillatory behavior of inequalities (3.20) and (3.21).



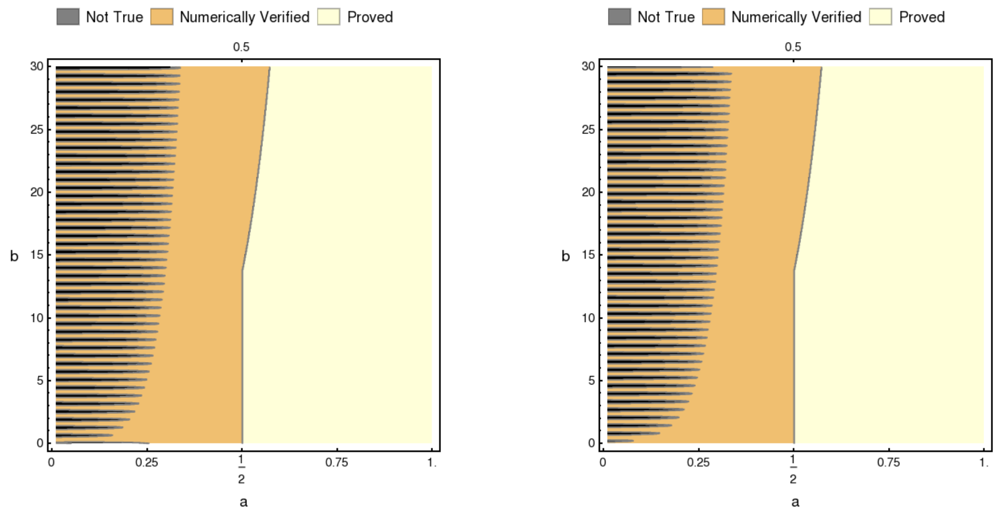
(a) (b)

Figure 4.6: For $0 \leq a \leq 1$ and $0 \leq b \leq 1000$, the regions where the monotonicity of fractional Fourier coefficients is proved, the monotonicity is numerically verified and the monotonicity does not hold.



(a) (b)

Figure 4.7: For $0 \leq a \leq 1$ and $800 \leq b \leq 1800$, the regions where the monotonicity of fractional Fourier coefficients is proved, the monotonicity is numerically verified and the monotonicity does not hold.



(a) (b)

Figure 4.8: For $0 \leq a \leq 1$ and $0 \leq b \leq 30$, the regions where the monotonicity of fractional Fourier coefficients is proved, the monotonicity is numerically verified and the monotonicity does not hold.

4.3.2 Fractional Fourier Coefficients of Large Degree

In Theorem 4.3.1, we have proved that the fractional Fourier coefficients of zero degree are monotone on domain $\mathcal{D}$. Now we show that the fractional Fourier coefficients $\left|\hat{f}_\alpha(k)\right|$ for $|k| > N(\gamma, \mu)$ are monotonic increasing on $\alpha > \alpha^*$ whereas monotonic decreasing on $\alpha_1 \leq \alpha < \alpha^*$, where $0 < \alpha_1 < \alpha^*$.

Theorem 4.3.2. *Let*

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ and $\mu \in \mathbb{R}^+$. Let $(\gamma, \mu) \in \mathcal{D}$ given in (4.28). Then there exists $N(\gamma, \mu) \in \mathbb{N}_0$ and $\alpha_1 = \alpha_1(\gamma, \mu)$ such that for all $|k| > N(\gamma, \mu)$

$$\left|\hat{f}_{\alpha_R}(k)\right| < \left|\hat{f}_{\beta_R}(k)\right| \quad \textit{for} \quad \alpha^* < \alpha_R < \beta_R \leq \frac{\pi}{2}$$

and

$$\left|\hat{f}_{\beta_L}(k)\right| > \left|\hat{f}_{\alpha_L}(k)\right| \quad \textit{for} \quad \alpha_1 \leq \beta_L < \alpha_L < \alpha^*,$$

where $0 < \alpha_1 < \alpha^*$ and α^* is the optimal fractional Fourier parameter.

Proof. Let $\left|\hat{f}_\alpha(k)\right|^2$ be denoted by $F_k(\alpha)$. From Lemma 4.1.1, we have

$$F_k(\alpha) = \frac{e^{\frac{-\pi\gamma k^2}{2\left(\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2\right)}}}{16\pi\sqrt{\pi^2\gamma^2+\frac{1}{4}(2\pi\mu-\cot\alpha)^2}} \times \left|\operatorname{erf}\left(\frac{\mathrm{i}\,(k-\pi(2\pi\mu-\cot\alpha))-2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}, \frac{\mathrm{i}\,(k+\pi(2\pi\mu-\cot\alpha))+2\pi^2\gamma}{2\sqrt{\pi\gamma+\frac{\mathrm{i}}{2}(2\pi\mu-\cot\alpha)}}\right)\right|^2. \tag{4.46}$$

The necessary and sufficient condition of $F_k(\alpha)$ to be monotone for all $(\gamma,\mu)\in\mathcal{D}$ is

$$\begin{aligned} F_k'(\alpha) < 0 \quad &\text{on} \quad \alpha^* < \alpha \le \frac{\pi}{2}, \\ F_k'(\alpha) > 0 \quad &\text{on} \quad \alpha_1 \le \alpha < \alpha^*, \end{aligned} \tag{4.47}$$

where we choose

$$\alpha_1 = \arctan\left(\frac{1}{2\pi\mu+\frac{\sqrt{\gamma}}{8}\left(\lambda e^{\pi^3\gamma}+\sqrt{\left(\lambda e^{a\pi^3\gamma}\right)^2-64\pi^2\gamma}\right)}\right).$$

We simplify (4.46) using the substitution

$$\pi\gamma = a(\gamma) \quad \text{and} \quad \frac{1}{2}(2\pi\mu-\cot\alpha) = b(\alpha). \tag{4.48}$$

For the sake of convenience, we write a and b instead of $a(\gamma)$ and $b(\alpha)$. When $\alpha^* < \alpha \le \frac{\pi}{2}$, then $0 < b \le \pi\mu$. The domain $\mathcal{D}$ in (4.28) becomes Ω_R as defined in (3.15)

$$\Omega_R = \left\{(a,b) : a \ge \frac{1}{2},\ 0 < b \le \frac{\sqrt{a\pi}}{8\pi}\left(\lambda e^{a\pi^2}+\sqrt{\left(\lambda e^{a\pi^2}\right)^2-64a\pi}\right)\right\},$$

where λ is given in (3.16). When $\alpha_1 \le \alpha < \alpha^*$, the domain $\mathcal{D}$ in (4.28) becomes Ω_L, where

$$\Omega_L = \left\{(a,b) : a \ge \frac{1}{2},\ \frac{-\sqrt{a\pi}}{8\pi}\left(\lambda e^{a\pi^2}+\sqrt{\left(\lambda e^{a\pi^2}\right)^2-64a\pi}\right) \le b < 0\right\}.$$

The method to compute the limits of Ω_L using α_1 can be found in Theorem 4.3.1. The necessary and sufficient condition (4.47) is equivalent to

$$\begin{aligned} G_k'(b) > 0 \quad &\text{for all} \quad (a,b)\in\Omega_R, \\ G_k'(b) < 0 \quad &\text{for all} \quad (a,b)\in\Omega_L. \end{aligned}$$

By using the substitution (4.48), the function $F_k(\alpha)$ in (4.46) becomes $G_k(b)$, where

$$G_k(b) = \frac{e^{\frac{-ak^2}{2(a^2+b^2)}}}{16\pi\sqrt{a^2+b^2}} \left| \operatorname{erf}\left(\frac{\mathrm{i}\,(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}\,(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \right|^2. \tag{4.49}$$

From Lemma 3.5.2, we write

$$\begin{aligned} &\left| \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \right|^2 \\ &= \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \\ &\times \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b) - 2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b) + 2\pi a}{2\sqrt{a-\mathrm{i}b}} \right). \end{aligned} \tag{4.50}$$

Using (4.50), we write $G_k(b)$ in (4.49) as

$$\begin{aligned} G_k(b) = \frac{e^{\frac{-ak^2}{2(a^2+b^2)}}}{16\pi\sqrt{a^2+b^2}} &\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \\ \times &\operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b) - 2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b) + 2\pi a}{2\sqrt{a-\mathrm{i}b}} \right). \end{aligned}$$

The domain Ω_R

We shall show that $G_k'(b) > 0$ for all $(a,b) \in \Omega_R$ when $|k| > N_0(a,b)$. The derivative of $G_k(b)$ with respect to b is calculated by using the derivative of the generalized error function in Lemma 3.5.4

$$\begin{aligned} G_k'(b) =& \frac{b\left(ak^2 - (a^2+b^2)\right) e^{\frac{-ak^2}{2(a^2+b^2)}}}{16\pi\sqrt{a^2+b^2}(a^2+b^2)^2} \\ &\times \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \\ &\times \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b) - 2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b) + 2\pi a}{2\sqrt{a-\mathrm{i}b}} \right) \\ &+ \frac{\sqrt{a-\mathrm{i}b}(a+\mathrm{i}b)^2 e^{\frac{-ak^2}{2(a^2+b^2)}}}{32\pi^{\frac{3}{2}}(a^2+b^2)^{\frac{5}{2}}} \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \\ &\times \left(\left((k-2\pi b) - \mathrm{i}2\pi a\right) e^{\frac{-(\mathrm{i}(k+2\pi b) - 2\pi a)^2}{4(a-\mathrm{i}b)}} - \left((k+2\pi b) + \mathrm{i}2\pi a\right) e^{\frac{-(\mathrm{i}(k-2\pi b) + 2\pi a)^2}{4(a-\mathrm{i}b)}} \right) \\ &+ \frac{\sqrt{a+\mathrm{i}b}(a-\mathrm{i}b)^2 e^{\frac{-ak^2}{2(a^2+b^2)}}}{32\pi^{\frac{3}{2}}(a^2+b^2)^{\frac{5}{2}}} \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b) - 2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b) + 2\pi a}{2\sqrt{a-\mathrm{i}b}} \right) \\ &\times \left(\left((k-2\pi b) + \mathrm{i}2\pi a\right) e^{\frac{-(\mathrm{i}(k+2\pi b) + 2\pi a)^2}{4(a+\mathrm{i}b)}} - \left((k+2\pi b) - \mathrm{i}2\pi a\right) e^{\frac{-(\mathrm{i}(k-2\pi b) - 2\pi a)^2}{4(a+\mathrm{i}b)}} \right). \end{aligned}$$

Taking

$$\frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{32\pi^{\frac{3}{2}}(a^2+b^2)^{\frac{5}{2}}}$$

common from all expression gives

$$\begin{aligned}
G_k'(b) =& \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{32\pi^{\frac{3}{2}}(a^2+b^2)^{\frac{5}{2}}}\Bigg[2\sqrt{\pi}b\left(ak^2-(a^2+b^2)\right)\\
&\times \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)\\
&\times \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)-2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b)+2\pi a}{2\sqrt{a-\mathrm{i}b}}\right)\\
&+\sqrt{a-\mathrm{i}b}(a+\mathrm{i}b)^2 \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)\\
&\times\left(((k-2\pi b)-\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{-(\mathrm{i}(k+2\pi b)-2\pi a)^2}{4(a-\mathrm{i}b)}}-((k+2\pi b)+\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{-(\mathrm{i}(k-2\pi b)+2\pi a)^2}{4(a-\mathrm{i}b)}}\right)\\
&+\sqrt{a+\mathrm{i}b}(a-\mathrm{i}b)^2 \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)-2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b)+2\pi a}{2\sqrt{a-\mathrm{i}b}}\right)\\
&\times\left(((k-2\pi b)+\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{-(\mathrm{i}(k+2\pi b)+2\pi a)^2}{4(a+\mathrm{i}b)}}-((k+2\pi b)-\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{-(\mathrm{i}(k-2\pi b)-2\pi a)^2}{4(a+\mathrm{i}b)}}\right)\Bigg].
\end{aligned}$$

Using (4.50), we write the above equation as

$$\begin{aligned}
G_k'(b) =& \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{32\pi^{\frac{3}{2}}(a^2+b^2)^{\frac{5}{2}}}\Bigg[2\sqrt{\pi}b\left(ak^2-(a^2+b^2)\right)\\
&\times \left|\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)\right|^2\\
&+\sqrt{a-\mathrm{i}b}(a+\mathrm{i}b)^2 \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)\\
&\times\left(((k-2\pi b)-\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{-(\mathrm{i}(k+2\pi b)-2\pi a)^2}{4(a-\mathrm{i}b)}}-((k+2\pi b)+\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{-(\mathrm{i}(k-2\pi b)+2\pi a)^2}{4(a-\mathrm{i}b)}}\right)\\
&+\sqrt{a+\mathrm{i}b}(a-\mathrm{i}b)^2 \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)-2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b)+2\pi a}{2\sqrt{a-\mathrm{i}b}}\right)\\
&\times\left(((k-2\pi b)+\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{-(\mathrm{i}(k+2\pi b)+2\pi a)^2}{4(a+\mathrm{i}b)}}-((k+2\pi b)-\mathrm{i}2\pi a)\,\mathrm{e}^{\frac{-(\mathrm{i}(k-2\pi b)-2\pi a)^2}{4(a+\mathrm{i}b)}}\right)\Bigg].
\end{aligned} \tag{4.51}$$

To prove $G_k'(b) > 0$ for all $(a,b) \in \Omega_R$ when $|k| > N_0(a,b)$, we expand $G_k'(b)$ using the asymptotic expansion of the generalized error functions. Let z_1, z_2, z_3 and z_4 be given in

(3.49). From Lemma 3.6.5 and Lemma 3.6.6, using the asymptotic expansion of

$$\operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b)-2\pi a}{2\sqrt{a+\mathrm{i}b}},\frac{\mathrm{i}(k+2\pi b)+2\pi a}{2\sqrt{a+\mathrm{i}b}}\right)=\operatorname{erf}(z_1,z_2)$$

and

$$\operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b)-2\pi a}{2\sqrt{a-\mathrm{i}b}},\frac{\mathrm{i}(k-2\pi b)+2\pi a}{2\sqrt{a-\mathrm{i}b}}\right)=\operatorname{erf}(z_3,z_4)$$

respectively, the derivative $G_k'(b)$ has an expansion of the form

$$\begin{aligned}
G_k'(b)=&\frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{16\pi^2(a^2+b^2)^2}\Bigg[4b\left(ak^2-(a^2+b^2)\right)\\
&\times\left(\frac{\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2a\pi)^3}+\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_2)\right.\\
&\left.-\frac{\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)^3}-\frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_1)\right)\\
&\times\left(\frac{\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)^3}+\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_3)\right.\\
&\left.-\frac{\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)^3}-\frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_4)\right)\\
&+\mathrm{i}(a+\mathrm{i}b)\left(((k-2\pi b)-\mathrm{i}2\pi a)\,\mathrm{e}^{-z_3^2}-((k+2\pi b)+\mathrm{i}2\pi a)\,\mathrm{e}^{-z_4^2}\right)\\
&\times\left(\frac{\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2a\pi)^3}+\frac{\mathrm{e}^{-z_2^2}}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_2)\right.\\
&\left.-\frac{\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)^3}-\frac{\mathrm{e}^{-z_1^2}}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_1)\right)\\
&-\mathrm{i}(a-\mathrm{i}b)\left(((k-2\pi b)+\mathrm{i}2\pi a)\,\mathrm{e}^{-z_1^2}-((k+2\pi b)-\mathrm{i}2\pi a)\,\mathrm{e}^{-z_2^2}\right)\\
&\times\left(\frac{\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)^3}+\frac{\mathrm{e}^{-z_3^2}}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_3)\right.\\
&\left.-\frac{\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)^3}-\frac{\mathrm{e}^{-z_4^2}}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_4)\right)\Bigg].
\end{aligned}$$

Multiplying the expressions on the right side of $G_k'(b)$ and using

$$\begin{aligned}
\mathrm{e}^{-z_1^2}\mathrm{e}^{-z_3^2}&=\mathrm{e}^{\frac{ak^2}{2(a^2+b^2)}}\mathrm{e}^{-2a\pi^2}\mathrm{e}^{\mathrm{i}2k\pi}\\
\mathrm{e}^{-z_1^2}\mathrm{e}^{-z_4^2}&=\mathrm{e}^{\frac{ak^2}{2(a^2+b^2)}}\mathrm{e}^{-2a\pi^2}
\end{aligned}$$

$$\mathrm{e}^{-z_2^2}\mathrm{e}^{-z_3^2} = \mathrm{e}^{\frac{ak^2}{2(a^2+b^2)}}\mathrm{e}^{-2a\pi^2}$$
$$\mathrm{e}^{-z_2^2}\mathrm{e}^{-z_4^2} = \mathrm{e}^{\frac{ak^2}{2(a^2+b^2)}}\mathrm{e}^{-2a\pi^2}\mathrm{e}^{-\mathrm{i}2k\pi},$$

we write $G'_k(b)$ as a sum of approximation term and the remainder term

$$G'_k(b) = \widetilde{G'}_k(b) + \theta_k(b),$$

where the approximation term $\widetilde{G'}_k(b)$ is given by

$$\begin{aligned}
\widetilde{G'}_k(b) =& \frac{\mathrm{e}^{-2a\pi^2}}{16\pi^2(a^2+b^2)^2}\Bigg[4b\left(ak^2-(a^2+b^2)\right) \\
&\times\Bigg(\frac{\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k+2\pi b)-\mathrm{i}2\pi a)^3\,((k+2\pi b)+\mathrm{i}2\pi a)^3} \\
&- \frac{\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\mathrm{e}^{-\mathrm{i}2k\pi}}{((k+2\pi b)-\mathrm{i}2\pi a)^3\,((k-2\pi b)-\mathrm{i}2\pi a)^3} \\
&- \frac{\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\mathrm{e}^{\mathrm{i}2k\pi}}{((k-2\pi b)+\mathrm{i}2\pi a)^3\,((k+2\pi b)+\mathrm{i}2\pi a)^3} \\
&+ \frac{\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k-2\pi b)+\mathrm{i}2\pi a)^3\,((k-2\pi b)-\mathrm{i}2\pi a)^3}\Bigg) \\
&+ \frac{\mathrm{i}(a+\mathrm{i}b)^2}{((k+2\pi b)-\mathrm{i}2\pi a)^3}\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)((k-2\pi b)-\mathrm{i}2\pi a) \\
&- \frac{\mathrm{i}(a+\mathrm{i}b)^2\mathrm{e}^{-\mathrm{i}2k\pi}}{((k+2\pi b)-\mathrm{i}2\pi a)^3}\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a) \\
&- \frac{\mathrm{i}(a+\mathrm{i}b)^2\mathrm{e}^{\mathrm{i}2k\pi}}{((k-2\pi b)+\mathrm{i}2\pi a)^3}\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)((k-2\pi b)-\mathrm{i}2\pi a) \\
&+ \frac{\mathrm{i}(a+\mathrm{i}b)^2}{((k-2\pi a)+\mathrm{i}2\pi a)^3}\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a) \\
&- \frac{\mathrm{i}(a-\mathrm{i}b)^2}{((k+2\pi b)+\mathrm{i}2\pi a)^3}\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k-2\pi b)+\mathrm{i}2\pi a) \\
&+ \frac{\mathrm{i}(a-\mathrm{i}b)^2\mathrm{e}^{\mathrm{i}2k\pi}}{((k+2\pi b)+\mathrm{i}2\pi a)^3}\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k+2\pi b)-\mathrm{i}2\pi a) \\
&+ \frac{\mathrm{i}(a-\mathrm{i}b)^2\mathrm{e}^{-\mathrm{i}2k\pi}}{((k-2\pi b)-\mathrm{i}2\pi a)^3}\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k-2\pi b)+\mathrm{i}2\pi a) \\
&- \frac{\mathrm{i}(a-\mathrm{i}b)^2}{((k-2\pi b)-\mathrm{i}2\pi a)^3}\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+(2a-\mathrm{i}b)\right)((k+2\pi b)-\mathrm{i}2\pi a)\Bigg],
\end{aligned} \tag{4.52}$$

and the remainder term $\theta_k(b)$ is given by

$$\begin{aligned}
\theta_k(b) =& \frac{\mathrm{e}^{-2\pi^2}}{16\pi^2(a^2+b^2)^2}\Bigg[4b\left(ak^2-(a^2+b^2)\right)\\
&\times\Bigg(\frac{\varepsilon_2(z_3)}{((k+2\pi b)+\mathrm{i}2\pi a)\left((k+2\pi b)-\mathrm{i}2\pi a\right)^3}\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\\
&-\frac{\mathrm{e}^{\mathrm{i}2k\pi}\varepsilon_2(z_3)}{((k-2\pi b)+\mathrm{i}2\pi a)^3\left((k+2\pi b)+\mathrm{i}2\pi a\right)}\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\\
&-\frac{\mathrm{e}^{-\mathrm{i}2k\pi}\varepsilon_2(z_4)}{((k+2\pi b)-\mathrm{i}2\pi a)^3\left((k-b\pi)-\mathrm{i}2a\pi\right)}\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\\
&+\frac{\varepsilon_2(z_4)}{((k-2\pi b)+\mathrm{i}2\pi a)^3\left((k-2\pi b)-\mathrm{i}2\pi a\right)}\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\\
&+\frac{\varepsilon_2(z_2)}{((k+2\pi b)+\mathrm{i}2\pi a)^3\left((k+2\pi b)-\mathrm{i}2\pi a\right)}\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\\
&-\frac{\mathrm{e}^{-\mathrm{i}2k\pi}\varepsilon_2(z_2)}{((k-2\pi b)-\mathrm{i}2\pi a)^3\left((k+2\pi b)-\mathrm{i}2\pi a\right)}\left(((k-2\pi b)-\mathrm{i}2a\pi)^2+2(a-\mathrm{i}b)\right)\\
&-\frac{\mathrm{e}^{\mathrm{i}2k\pi}\varepsilon_2(z_1)}{((k+2\pi b)+\mathrm{i}2\pi a)^3\left((k-2\pi b)+\mathrm{i}2\pi a\right)}\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\\
&+\frac{\varepsilon_2(z_1)}{((k-2\pi b)-\mathrm{i}2\pi a)^3\left((k-2\pi b)+\mathrm{i}2\pi a\right)}\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)\\
&+\frac{\varepsilon_2(z_2)\varepsilon_2(z_3)}{(k+2\pi b)^2+4\pi^2a^2}-\frac{\mathrm{e}^{\mathrm{i}2k\pi}\varepsilon_2(z_1)\varepsilon_2(z_3)}{((k-2\pi b)+\mathrm{i}2\pi a)\left((k+2\pi b)+\mathrm{i}2\pi a\right)}\\
&-\frac{\mathrm{e}^{-\mathrm{i}2k\pi}\varepsilon_2(z_2)\varepsilon_2(z_4)}{((k+2\pi b)-\mathrm{i}2\pi a)\left((k-2\pi b)-\mathrm{i}2\pi a\right)}+\frac{\varepsilon_2(z_1)\varepsilon_2(z_4)}{(k-2\pi b)^2+4\pi^2a^2}\Bigg)\\
&+\frac{\mathrm{i}(a+\mathrm{i}b)^2\left((k-2\pi b)-\mathrm{i}2a\pi\right)}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_2)-\frac{\mathrm{e}^{-\mathrm{i}2k\pi}\mathrm{i}(a+\mathrm{i}b)^2\left((k+2\pi b)+\mathrm{i}2\pi a\right)}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_2)\\
&-\frac{\mathrm{e}^{\mathrm{i}2k\pi}\mathrm{i}(a+\mathrm{i}b)^2\left((k-2\pi b)-\mathrm{i}2\pi a\right)}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_1)+\frac{\mathrm{i}(a+\mathrm{i}b)^2\left((k+2\pi b)+\mathrm{i}2\pi a\right)}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_1)\\
&-\frac{\mathrm{i}(a-\mathrm{i}b)^2\left((k-2\pi b)+\mathrm{i}2\pi a\right)}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_3)+\frac{\mathrm{e}^{\mathrm{i}2k\pi}\mathrm{i}(a-\mathrm{i}b)^2\left((k+2\pi b)-\mathrm{i}2\pi a\right)}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_3)\\
&+\frac{\mathrm{i}(a-\mathrm{i}b)^2\left((k-2\pi b)+\mathrm{i}2\pi a\right)}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_4)-\frac{\mathrm{e}^{-\mathrm{i}2k\pi}\mathrm{i}(a-\mathrm{i}b)^2\left((k+2\pi b)-\mathrm{i}2\pi a\right)}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_4)\Bigg].
\end{aligned}\tag{4.53}$$

We show that $\widetilde{G'}_k(b)>0$. We simplify $\widetilde{G'}_k(b)$ in (4.52) by multiplying and dividing

$$((k+2\pi b)+\mathrm{i}2\pi a)^3\left((k+2\pi b)-\mathrm{i}2\pi a\right)^3\left((k-2\pi b)+\mathrm{i}2\pi a\right)^3\left((k-2\pi b)-\mathrm{i}2\pi a\right)^3,$$

we get

$$
\begin{aligned}
\widetilde{G'}_k(b) =& \frac{(a^2+b^2)^{-2}\mathrm{e}^{-2a\pi^2}}{16\pi^2\left((k+2\pi)^2+4\pi^2a^2\right)^3\left((k-2\pi b)^2+4\pi^2a^2\right)^3}\Bigg[4b\left(ak^2-(a^2+b^2)\right) \\
&\times\Bigg(\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right) \\
&\times((k-2\pi b)+\mathrm{i}2\pi a)^3((k-2\pi b)-\mathrm{i}2\pi a)^3 \\
&-\mathrm{e}^{-\mathrm{i}2k\pi}\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right) \\
&\times((k-2\pi b)+\mathrm{i}2\pi a)^3((k+2\pi b)+\mathrm{i}2\pi a)^3 \\
&-\mathrm{e}^{\mathrm{i}2k\pi}\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right) \\
&\times((k+2\pi b)-\mathrm{i}2\pi a)^3((k-2\pi b)-\mathrm{i}2\pi a)^3 \\
&+\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right) \\
&\times((k+2\pi b)-\mathrm{i}2\pi a)^3((k+2\pi b)+\mathrm{i}2\pi a)^3\Bigg) \\
&+\mathrm{i}(a+\mathrm{i}b)^2\Bigg(\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a)^3 \\
&\times((k-2\pi b)+\mathrm{i}2\pi a)^3((k-2\pi b)-\mathrm{i}2\pi a)^4 \\
&-\mathrm{e}^{-\mathrm{i}2k\pi}\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a)^4 \\
&\times((k-2\pi b)+\mathrm{i}2\pi a)^3((k-2\pi b)-\mathrm{i}2\pi a)^3 \\
&-\mathrm{e}^{\mathrm{i}2k\pi}\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a)^3 \\
&\times((k+2\pi b)-\mathrm{i}2\pi a)^3((k-2\pi b)-\mathrm{i}2\pi a)^4 \\
&+\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a)^4 \\
&\times((k+2\pi b)-\mathrm{i}2\pi a)^3((k-2\pi b)-\mathrm{i}2\pi a)^3\Bigg) \\
&+\mathrm{i}(a-\mathrm{i}b)^2\Bigg(-\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k+2\pi b)-\mathrm{i}2\pi a)^3 \\
&\times((k-2\pi b)-\mathrm{i}2\pi a)^3((k-2\pi b)+\mathrm{i}2\pi a)^4 \\
&+\mathrm{e}^{\mathrm{i}2k\pi}\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k+2\pi b)-\mathrm{i}2\pi a)^4 \\
&\times((k-2\pi b)+\mathrm{i}2\pi a)^3((k-2\pi b)-\mathrm{i}2\pi a)^3 \\
&+\mathrm{e}^{-\mathrm{i}2k\pi}\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a)^3 \\
&\times((k+2\pi b)-\mathrm{i}2\pi a)^3((k-2\pi b)+\mathrm{i}2\pi a)^4 \\
&-\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a)^3((k-2\pi b)+\mathrm{i}2\pi a)^3 \\
&\times((k+2\pi b)-\mathrm{i}2\pi a)^4\Bigg)\Bigg].
\end{aligned}
$$

We obtain the following expression after simplification

$$
\begin{aligned}
\widetilde{G'}_k(b) =& \frac{(a^2+b^2)^{-2}\mathrm{e}^{-2a\pi^2}}{16\pi^2\left((k+2\pi)^2+4\pi^2a^2\right)^3\left((k-2\pi b)^2+4\pi^2a^2\right)^3} \\
&\times 32b\left(a^2+b^2\right)^2\Bigg(\left(6\pi^2k^8+(64(a^2-b^2)\pi^4+4a\pi^2-1)\right)k^6 \\
&+4\pi^2\left(16(3a^4-2a^2b^2+3b^4)\pi^4+12a(a^2+5b^2)\pi^2-3(a^2+5b^2)\right)k^4 \\
&+48(a^2+b^2)(a^2+5b^2)(4a\pi^2-1)\pi^4k^2 \\
&-64(a^2+b^2)^3(8(a^2+b^2)\pi^4-4a\pi^2+1)\pi^6 \\
&-\cos(2k\pi)(2\pi^2k^8-(32(a^2+b^2)\pi^4+44a\pi^2+1)k^6 \\
&+4(16(-5a^4+6a^2b^2+3b^4)\pi^4+4a(25a^2+21b^2)\pi^2+3(5a^2+b^2))\pi^2k^4 \\
&-16(a^2+b^2)(32(a^2+b^2)^2\pi^4-12a(a^2-3b^2)\pi^2+3(5a^2+b^2))\pi^4k^2 \\
&+64(a^2+b^2)^3(8(a^2+b^2)\pi^4-4a\pi^2+1)\pi^6) \\
&-4\pi\sin(2k\pi)((4a\pi^2+1)k^7-(16a(a^2-3b^2)\pi^4+12(4a^2+b^2)\pi^2+3a)k^5 \\
&+8(8a(a^2-3b^2)(a^2+b^2)\pi^6-2(5a^4+12a^2b^2+3b^4)\pi^4-a(5a^2+3b^2)\pi^2)k^3 \\
&-16(16a(a^2+b^2)^3\pi^8-4(a^2+b^2)\left(2a^4+a^2b^2-b^4\right)\pi^6+3a(a^2+b^2)^2\pi^4)k)\Bigg).
\end{aligned}
$$

Using $\cos(2k\pi)=1$ and $\sin(2k\pi)=1$ for all $k\in\mathbb{Z}$, the above expression simplifies to

$$
\begin{aligned}
\widetilde{G'}_k(b) =& \frac{8b\mathrm{e}^{-2a\pi^2}}{\pi^2\left((k+2\pi b)^2+4\pi^2a^2\right)^3\left((k-2\pi b)^2+4\pi^2a^2\right)^3} \\
&\times\Bigg(k^8+4\left(2(3a^2-b^2)\pi^2+3a\right)k^6 \\
&+2\left(64a^2(a^2-b^2)\pi^4-4a(11a^2+3b^2)\pi^2-9(a^2+b^2)\right)k^4 \\
&+8\left(16b^2(a^2+b^2)^3\pi^6+48ab^2(a^2+b^2)\pi^4+6(a^4-b^4)\pi^2\right)k^2 \\
&-16(a^2+b^2)^3\left(8(2(a^2+b^2)-a)\pi^2+2\right)\pi^4\Bigg).
\end{aligned}
$$

We write $\widetilde{G'}_k(b)$ in order notation written as

$$
\widetilde{G'}_k(b)=\frac{8b\mathrm{e}^{-2a\pi^2}}{\pi^2}\mathcal{O}(k^{-4}). \tag{4.54}
$$

Now we find an estimate of the remainder $\theta_k(b)$ given in (4.53). Using $\mathrm{e}^{-\mathrm{i}2k\pi} = \mathrm{e}^{\mathrm{i}2k\pi} = 1$ for all $k \in \mathbb{Z}$, and arranging like terms in (4.53), we write

$$\begin{aligned}
\theta_k(b) =& \frac{\mathrm{e}^{-2a\pi^2}}{4\pi^2(a^2+b^2)^2} \\
&\times \Bigg[\Bigg(\frac{b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k-2\pi b)-\mathrm{i}2\pi a)^3\,((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_1) \\
&- \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k+2\pi b)+\mathrm{i}2\pi a)^3\,((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_1) \\
&- \frac{\pi(a+\mathrm{i}b)(a^2+b^2)}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_1) \\
&+ \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k+2\pi b)+\mathrm{i}2\pi a)^3\,((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_2) \\
&- \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k-2\pi b)-\mathrm{i}2\pi a)^3\,((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_2) \\
&+ \frac{\pi(a+\mathrm{i}b)(a^2+b^2)}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_2) \\
&+ \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)}{((k+2\pi b)-\mathrm{i}2\pi a)^3\,((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_3) \\
&- \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)}{((k-2\pi b)+\mathrm{i}2\pi a)^3\,((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_3) \\
&+ \frac{\pi(a-\mathrm{i}b)(a^2+b^2)}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_3) \\
&+ \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)}{((k-2\pi b)+\mathrm{i}2\pi a)^3\,((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_4) \\
&- \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)}{((k+2\pi b)-\mathrm{i}2\pi a)^3\,((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_4) \\
&- \frac{\pi(a-\mathrm{i}b)(a^2+b^2)}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_4) \Bigg) \\
&+ b\left(ak^2-(a^2+b^2)\right) \\
&\times \Bigg(\frac{\varepsilon_2(z_2)\varepsilon_2(z_3)}{(k+2\pi b)^2+4\pi^2a^2} - \frac{\varepsilon_2(z_1)\varepsilon_2(z_3)}{((k-2\pi b)+\mathrm{i}2\pi a)\,((k+2\pi b)+\mathrm{i}2\pi a)} \\
&- \frac{\varepsilon_2(z_2)\varepsilon_2(z_4)}{((k+2\pi b)-\mathrm{i}2\pi a)\,((k-2\pi b)-\mathrm{i}2\pi a)} + \frac{\varepsilon_2(z_1)\varepsilon_2(z_4)}{(k-2\pi b)^2+4\pi^2a^2} \Bigg) \Bigg].
\end{aligned}$$

The remainder term $\theta_k(b)$ can be written as

$$\theta_k(b) = \frac{\mathrm{e}^{-2a\pi^2}}{4\pi^2(a^2+b^2)^2}\left(\delta_{1,k}(a,b) + \delta_{2,k}(a,b) + \delta_{3,k}(a,b) + \delta_{4,k}(a,b) + \Delta_k(a,b)\right), \quad (4.55)$$

where $\delta_{1,k}(a,b)$, $\delta_{2,k}(a,b)$, $\delta_{3,k}(a,b)$, $\delta_{4,k}(a,b)$ and $\Delta_k(a,b)$ are given by

$$\begin{aligned}\delta_{1,k}(a,b) =& \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k-2\pi b)-\mathrm{i}2\pi a)^3\left((k-2\pi b)+\mathrm{i}2\pi a\right)}\varepsilon_2(z_1)\\ &-\frac{b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k+2\pi b)+\mathrm{i}2\pi a)^3\left((k-2\pi b)+\mathrm{i}2\pi a\right)}\varepsilon_2(z_1)\\ &-\frac{\pi(a+\mathrm{i}b)(a^2+b^2)}{((k-2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_1),\end{aligned} \quad (4.56)$$

$$\begin{aligned}\delta_{2,k}(a,b) =& \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k+2\pi b)+\mathrm{i}2\pi a)^3\left((k+2\pi b)-\mathrm{i}2\pi a\right)}\varepsilon_2(z_2)\\ &-\frac{b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)}{((k-2\pi b)-\mathrm{i}2\pi a)^3\left((k+2\pi b)-\mathrm{i}2\pi a\right)}\varepsilon_2(z_2)\\ &+\frac{\pi(a+\mathrm{i}b)(a^2+b^2)}{((k+2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_2),\end{aligned} \quad (4.57)$$

$$\begin{aligned}\delta_{3,k}(a,b) =& \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)}{((k+2\pi b)-\mathrm{i}2\pi a)^3\left((k+2\pi b)+\mathrm{i}2\pi a\right)}\varepsilon_2(z_3)\\ &-\frac{b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)}{((k-2\pi b)+\mathrm{i}2\pi a)^3\left((k+2\pi b)+\mathrm{i}2\pi a\right)}\varepsilon_2(z_3)\\ &+\frac{\pi(a-\mathrm{i}b)(a^2+b^2)}{((k+2\pi b)+\mathrm{i}2\pi a)}\varepsilon_2(z_3),\end{aligned} \quad (4.58)$$

$$\begin{aligned}\delta_{4,k}(a,b) =& \frac{b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)+\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)}{((k-2\pi b)+\mathrm{i}2\pi a)^3\left((k-2\pi b)-\mathrm{i}2\pi a\right)}\varepsilon_2(z_4)\\ &-\frac{b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)-\mathrm{i}2\pi a)^2+2(a+\mathrm{i}b)\right)}{((k+2\pi b)-\mathrm{i}2\pi a)^3\left((k-2\pi b)-\mathrm{i}2\pi a\right)}\varepsilon_2(z_4)\\ &-\frac{\pi(a-\mathrm{i}b)(a^2+b^2)}{((k-2\pi b)-\mathrm{i}2\pi a)}\varepsilon_2(z_4)\end{aligned} \quad (4.59)$$

and

$$\begin{aligned}\Delta_k(a,b) =& b\left(ak^2-(a^2+b^2)\right)\\ &\times\left(\frac{\varepsilon_2(z_2)\varepsilon_2(z_3)}{(k+2\pi b)^2+4\pi^2a^2}-\frac{\varepsilon_2(z_1)\varepsilon_2(z_3)}{((k-2\pi b)+\mathrm{i}2\pi a)\,((k+2\pi b)+\mathrm{i}2\pi a)}\right.\\ &\left.-\frac{\varepsilon_2(z_2)\varepsilon_2(z_4)}{((k+2\pi b)-\mathrm{i}2\pi a)\,((k-2\pi b)-\mathrm{i}2\pi a)}+\frac{\varepsilon_2(z_1)\varepsilon_2(z_4)}{(k-2\pi b)^2+4\pi^2a^2}\right)\end{aligned}\tag{4.60}$$

respectively. The estimate for remainder (4.55) is given by

$$|\theta_k(b)|\le\frac{\mathrm{e}^{-2a\pi^2}}{4\pi^2(a^2+b^2)^2}\left(|\delta_{1,k}(a,b)|+|\delta_{2,k}(a,b)|+|\delta_{3,k}(a,b)|+|\delta_{4,k}(a,b)|+|\Delta_k(a,b)|\right).\tag{4.61}$$

To estimate of the remainder in (4.61), we need $|\delta_{1,k}(a,b)|$, $|\delta_{2,k}(a,b)|$, $|\delta_{3,k}(a,b)|$, $|\delta_{4,k}(a,b)|$ and $|\Delta_k(a,b)|$. First, we take $\delta_{1,k}(a,b)$ in (4.56) and write $\delta_{1,k}(a,b)$ as a single fraction

$$\delta_{1,k}(a,b)=\frac{Q_{1,k}(a,b)}{((k-2\pi b)+\mathrm{i}2\pi a)\,((k-2\pi b)-\mathrm{i}2\pi a)^3\,((k+2\pi b)+\mathrm{i}2\pi a)^3}\varepsilon_2(z_1),\tag{4.62}$$

where

$$\begin{aligned}Q_{1,k}(a,b) =& b\left(ak^2-(a^2+b^2)\right)\left(((k-2\pi b)-\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k+2\pi b)+\mathrm{i}2\pi a)^3\\ &-b\left(ak^2-(a^2+b^2)\right)\left(((k+2\pi b)+\mathrm{i}2\pi a)^2+2(a-\mathrm{i}b)\right)((k-2\pi b)-\mathrm{i}2\pi a)^3\\ &-\pi(a+\mathrm{i}b)(a^2+b^2)\,((k-2\pi b)-\mathrm{i}2\pi a)^3\,((k+2\pi b)+\mathrm{i}2\pi a)^3.\end{aligned}$$

After simplifications, we get

$$\begin{aligned}Q_{1,k}(a,b) =& -\pi(a-\mathrm{i}b)\big[\,((k+2\pi b)+\mathrm{i}2\pi a)^2\,((k-2\pi b)-\mathrm{i}2\pi a)^2\\ &\times\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2-\mathrm{i}2b(ak^2-2(a^2+b^2))\right)\\ &+\mathrm{i}8\pi b(a+\mathrm{i}b)^2\left(ak^2-(a^2+b^2)\right)\left(3k^2-4(a^2-b^2)\pi^2+\mathrm{i}8ab\pi^2\right)\big].\end{aligned}$$

Now from (4.62) we calculate the absolute value of $\delta_{1,k}(a,b)$

$$|\delta_{1,k}(a,b)|=\frac{|Q_{1,k}(a,b)|}{\left((k-2\pi b)^2+4\pi^2a^2\right)^2\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}|\varepsilon_2(z_1)|.\tag{4.63}$$

The absolute value of $Q_{1,k}(a,b)$ can be estimated as

$$\begin{aligned}|Q_{1,k}(a,b)| \le& \pi\sqrt{a^2+b^2}\Bigg[\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right)\\ &\times\left(\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2\right)^2+4b^2\left(ak^2-2(a^2+b^2)\right)^2\right)^{\frac{1}{2}}\\ &+8\pi b\left(a^2+b^2\right)\left(4ak^2-(4a^2+b^2)\right)\left(\left(3k^2-4(a^2-b^2)\pi^2\right)^2+64a^2b^2\pi^4\right)^{\frac{1}{2}}\Bigg]\\ =&\pi\sqrt{a^2+b^2}C_k(a,b),\end{aligned}$$

where

$$\begin{aligned} C_k(a,b) = &\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right) \\ &\times\left(\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2\right)^2+4b^2\left(ak^2-2(a^2+b^2)\right)^2\right)^{\frac{1}{2}} \\ &+8\pi b\left(a^2+b^2\right)\left(4ak^2-(4a^2+b^2)\right)\left(\left(3k^2-4(a^2-b^2)\pi^2\right)^2+64a^2b^2\pi^4\right)^{\frac{1}{2}}. \end{aligned} \tag{4.64}$$

The remainder term $\varepsilon_2(z_1)$ has an estimate

$$|\varepsilon_2(z_1)| \leq \frac{12(a^2+b^2)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^2}.$$

Therefore, replacing $|Q_{1,k}(a,b)|$ and $|\varepsilon_2(z_1)|$ in (4.63), we obtain the estimate for $\delta_{1,k}(a,b)$

$$|\delta_{1,k}(a,b)| \leq \frac{12\pi(a^2+b^2)^{\frac{3}{2}}C_k(a,b)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^4\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}. \tag{4.65}$$

Now we take $\delta_{2,k}(a,b)$ in (4.57) and write $\delta_{2,k}(a,b)$ as a single fraction

$$\delta_{2,k}(a,b) = \frac{Q_{2,k}(a,b)}{\left((k+2\pi b)-\mathrm{i}2\pi a\right)\left((k-2\pi b)-\mathrm{i}2\pi a\right)^3\left((k+2\pi b)+\mathrm{i}2\pi a\right)^3}\varepsilon_2(z_2), \tag{4.66}$$

where

$$\begin{aligned} Q_{2,k}(a,b) =& b\left(ak^2-(a^2+b^2)\right)\left(\left((k-2\pi b)-\mathrm{i}2\pi a\right)^2+2(a-\mathrm{i}b)\right)\left((k+2\pi b)+\mathrm{i}2\pi a\right)^3 \\ &- b\left(ak^2-(a^2+b^2)\right)\left(\left((k+2\pi b)+\mathrm{i}2\pi b\right)^2+2(a-\mathrm{i}b)\right)\left((k-2\pi b)-\mathrm{i}2\pi a\right)^3 \\ &+ \pi(a+\mathrm{i}b)(a^2+b^2)\left((k-2\pi b)-\mathrm{i}2\pi a\right)^3\left((k+2\pi b)+\mathrm{i}2\pi a\right)^3. \end{aligned}$$

After simplifications, we get

$$\begin{aligned} Q_{2,k}(a,b) =& \pi(a-\mathrm{i}b)\big[\left((k+2\pi b)+\mathrm{i}2\pi a\right)^2\left((k-2\pi b)-\mathrm{i}2\pi a\right)^2 \\ &\times\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2-\mathrm{i}2b(ak^2-2(a^2+b^2))\right) \\ &-\mathrm{i}8\pi b(a-\mathrm{i}b)^2\left(ak^2-4(a^2+b^2)\right)\left(3k^2-4(a^2-b^2)\pi^2+\mathrm{i}8ab\pi^2\right)\big]. \end{aligned}$$

Now from (4.66) we calculate the absolute value of $\delta_{2,k}(a,b)$

$$|\delta_{2,k}(a,b)| = \frac{|Q_{2,k}(a,b)|}{\left((k+2\pi b)^2+4\pi^2a^2\right)^2\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}|\varepsilon_2(z_2)|. \tag{4.67}$$

The absolute value of $Q_{2,k}(a,b)$ can be estimated as

$$\begin{aligned} |Q_{2,k}(a,b)| \leq& \pi\sqrt{a^2+b^2}\Bigg[\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right) \\ &\times\left(\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2\right)^2+4b^2\left(ak^2-2(a^2+b^2)\right)^2\right)^{\frac{1}{2}} \\ &+8\pi b\left(a^2+b^2\right)\left(ak^2-(a^2+b^2)\right)\left(\left(3k^2-4(a^2-b^2)\pi^2\right)^2+64a^2b^2\pi^4\right)^{\frac{1}{2}}\Bigg] \\ =& \pi\sqrt{a^2+b^2}C_k(a,b), \end{aligned}$$

where $C_k(a,b)$ is given in (4.64). The remainder term $\varepsilon_2(z_2)$ has an estimate

$$|\varepsilon_2(z_2)| \leq \frac{12(a^2+b^2)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^2}.$$

Therefore, replacing $|Q_{2,k}(a,b)|$ and $|\varepsilon_2(z_2)|$ in (4.67), we obtain the estimate for $\delta_{2,k}(a,b)$

$$|\delta_{2,k}(a,b)| \leq \frac{12\pi(a^2+b^2)^{\frac{3}{2}}C_k(a,b)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^4\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}. \tag{4.68}$$

Now we simplify $\delta_{3,k}(a,b)$ in (4.58) and write $\delta_{3,k}(a,b)$ as a single fraction

$$\delta_{3,k}(a,b) = \frac{Q_{3,k}(a,b)}{\left((k+2\pi b)+\mathrm{i}2\pi a\right)\left((k+2\pi b)-\mathrm{i}2\pi a\right)^3\left((k-2\pi b)+\mathrm{i}2\pi a\right)^3}\varepsilon_2(z_3), \tag{4.69}$$

where

$$\begin{aligned}Q_{3,k}(a,b) =& b\left(ak^2-(a^2+b^2)\right)\left(\left((k+2\pi b)-\mathrm{i}2\pi a\right)^2+2(a+\mathrm{i}b)\right)\left((k-2\pi b)+\mathrm{i}2\pi a\right)^3\\ &- b\left(ak^2-(a^2+b^2)\right)\left(\left((k-2\pi b)+\mathrm{i}2\pi a\right)^2+2(a+\mathrm{i}b)\right)\left((k+2\pi b)-\mathrm{i}2\pi a\right)^3\\ &+\pi(a-\mathrm{i}b)(a^2+b^2)\left((k+2\pi b)-\mathrm{i}2\pi a\right)^3\left((k-2\pi b)+\mathrm{i}2\pi a\right)^3.\end{aligned}$$

After simplifications, we get

$$\begin{aligned}Q_{3,k}(a,b) =& \pi(a+\mathrm{i}b)\big[\left((k+2\pi b)-\mathrm{i}2\pi a\right)^2\left((k-2\pi b)+\mathrm{i}2\pi a\right)^2\\ &\times\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2+\mathrm{i}2b(ak^2-2(a^2+b^2))\right)\\ &+\mathrm{i}8\pi b(a+\mathrm{i}b)^2\left(ak^2-(a^2+b^2)\right)\left(3k^2-4(a^2-b^2)\pi^2-\mathrm{i}8ab\pi^2\right)\big].\end{aligned}$$

From (4.69), the absolute value of $\delta_{3,k}(a,b)$ is given by

$$|\delta_{3,k}(a,b)| \leq \frac{|Q_{3,k}(a,b)|}{\left((k+2\pi b)^2+4\pi^2a^2\right)^2\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}|\varepsilon_2(z_3)|. \tag{4.70}$$

The absolute value of $Q_{3,k}(a,b)$ can be estimated as

$$\begin{aligned}|Q_{3,k}(a,b)| \leq& \pi\sqrt{a^2+b^2}\Bigg[\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right)\\ &\times\left(\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2\right)^2+4b^2\left(ak^2-2(a^2+b^2)\right)^2\right)^{\frac{1}{2}}\\ &+8\pi b\left(a^2+b^2\right)\left(4ak^2-(4a^2+b^2)\right)\left(\left(3k^2-4(a^2-b^2)\pi^2\right)^2+64a^2b^2\pi^4\right)^{\frac{1}{2}}\Bigg]\\ =& \pi\sqrt{a^2+b^2}C_k(a,b),\end{aligned}$$

where $C_k(a,b)$ is given in (4.64). The remainder term $\varepsilon_2(z_3)$ has an estimate

$$|\varepsilon_2(z_3)| < \frac{12\sqrt{2}(a^2+b^2)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^2}.$$

Therefore, replacing $|Q_{3,k}(a,b)|$ and $|\varepsilon_2(z_3)|$ in (4.70), we obtain the estimate for $\delta_{3,k}(a,b)$

$$|\delta_{3,k}(a,b)| < \frac{12\sqrt{2}\pi(a^2+b^2)^{\frac{3}{2}}C_k(a,b)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^4\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}. \tag{4.71}$$

Now we consider $\delta_{4,k}(a,b)$ in (4.59) and write $\delta_{4,k}(a,b)$ as a single fraction

$$\delta_{4,k}(a,b) = \frac{Q_{4,k}(a,b)}{\left((k-2\pi b)-\mathrm{i}2\pi a\right)\left((k+2\pi b)-\mathrm{i}2\pi a\right)^3\left((k-2\pi b)+\mathrm{i}2\pi a\right)^3}\varepsilon_2(z_4), \tag{4.72}$$

where

$$\begin{aligned}
Q_{4,k}(a,b) =& b\left(ak^2-(a^2+b^2)\right)\left(\left((k-2\pi b)+\mathrm{i}2\pi a\right)^2+2(a+\mathrm{i}b)\right)\left((k+2\pi b)-\mathrm{i}2\pi a\right)^3 \\
&- b\left(ak^2-(a^2+b^2)\right)\left(\left((k+2\pi b)-\mathrm{i}2\pi a\right)^2+2(a+\mathrm{i}b)\right)\left((k-2\pi b)+\mathrm{i}2\pi a\right)^3 \\
&- \pi(a-\mathrm{i}b)(a^2+b^2)\left((k+2\pi b)-\mathrm{i}2\pi a\right)^3\left((k-2\pi a)+\mathrm{i}2\pi a\right)^3.
\end{aligned}$$

After simplifications, we get

$$\begin{aligned}
Q_{4,k}(a,b) =& -\pi(a+\mathrm{i}b)\Big[\left((k+2\pi b)-\mathrm{i}2\pi a\right)^2\left((k-2\pi b)+\mathrm{i}2\pi a\right)^2 \\
&\times\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2+\mathrm{i}2b(ak^2-2(a^2+b^2))\right) \\
&+\mathrm{i}8\pi b(a-\mathrm{i}b)^2\left(ak^2-(a^2+b^2)\right)\left(3k^2-4(a^2-b^2)\pi^2-\mathrm{i}8ab\pi^2\right)\Big].
\end{aligned}$$

From (4.72), the absolute value $\delta_{4,k}(a,b)$ can be estimated as

$$|\delta_{4,k}(a,b)| \le \frac{|Q_{4,k}(a,b)|}{\left((k-2\pi b)^2+4\pi^2a^2\right)^2\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}|\varepsilon_2(z_4)|. \tag{4.73}$$

The estimate of the absolute value of $Q_{4,k}(a,b)$ is

$$\begin{aligned}
|Q_{4,k}(a,b)| \le& \pi\sqrt{a^2+b^2}\Bigg[\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right) \\
&\times\left(\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2\right)^2+4b^2\left(ak^2-2(a^2+b^2)\right)^2\right)^{\frac{1}{2}} \\
&+8\pi b\left(a^2+b^2\right)\left(ak^2-(a^2+b^2)\right)\left(\left(3k^2-4(a^2-b^2)\pi^2\right)^2+64a^2b^2\pi^4\right)^{\frac{1}{2}}\Bigg] \\
=& \pi\sqrt{a^2+b^2}C_k(a,b),
\end{aligned}$$

$C_k(a,b)$ is given in (4.64). The remainder term $\varepsilon_2(z_4)$ has an estimate

$$|\varepsilon_2(z_4)| < \frac{12\sqrt{2}(a^2+b^2)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^2}.$$

Therefore, replacing $|Q_{4,k}(a,b)|$ and $|\varepsilon_2(z_4)|$ in (4.73), we obtain the estimate for $\delta_{4,k}(a,b)$

$$|\delta_{4,k}(a,b)| < \frac{12\sqrt{2}\pi(a^2+b^2)^{\frac{3}{2}}C_k(a,b)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^4\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}. \tag{4.74}$$

Finally we simplify $\Delta_k(a,b)$, where

$$\begin{aligned}\Delta_k(a,b) =& b\left(ak^2-(a^2+b^2)\right)\\ &\times\Bigg(-\frac{\varepsilon_2(z_1)\varepsilon_2(z_3)}{((k-2\pi b)+\mathrm{i}2\pi a)\,((k+2\pi b)+\mathrm{i}2\pi a)}\\ &-\frac{\varepsilon_2(z_2)\varepsilon_2(z_4)}{((k+2\pi b)-\mathrm{i}2\pi a)\,((k-2\pi b)-\mathrm{i}2\pi a)}\\ &+\frac{\varepsilon_2(z_1)\varepsilon_2(z_4)}{(k-2\pi b)^2+4\pi^2a^2}+\frac{\varepsilon_2(z_2)\varepsilon_2(z_3)}{(k+2\pi b)^2+4\pi^2a^2}\Bigg).\end{aligned}$$

The absolute value of $\Delta_k(a,b)$ in (4.60) is

$$\begin{aligned}|\Delta_k(a,b)| \le& b\left(ak^2-(a^2+b^2)\right)\\ &\times\Bigg(\frac{|\varepsilon_2(z_1)||\varepsilon_2(z_3)|}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\,((k-2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}}\\ &+\frac{|\varepsilon_2(z_2)||\varepsilon_2(z_4)|}{((k-2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}\,((k+2\pi b)^2+4\pi^2a^2)^{\frac{1}{2}}}\\ &+\frac{|\varepsilon_2(z_1)||\varepsilon_2(z_4)|}{((k-2\pi b)^2+4\pi^2a^2)}+\frac{|\varepsilon_2(z_2)||\varepsilon_2(z_3)|}{((k+2\pi b)^2+4\pi^2a^2)}\Bigg).\end{aligned}$$

Using the estimates of $\varepsilon_2(z_1)$, $\varepsilon_2(z_2)$, $\varepsilon_2(z_3)$ and $\varepsilon_2(z_4)$, we have

$$\begin{aligned}|\Delta_k(a,b)| <& b\left(ak^2-(a^2+b^2)\right)\\ &\times\Bigg(\frac{144\sqrt{2}(a^2+b^2)^2}{((k+2\pi b)^2+4\pi^2a^2)^{\frac{5}{2}}\,((k-2\pi b)^2+4\pi^2a^2)^{\frac{5}{2}}}\\ &+\frac{144\sqrt{2}(a^2+b^2)^2}{((k-2\pi b)^2+4\pi^2a^2)^{\frac{5}{2}}\,((k+2\pi b)^2+4\pi^2a^2)^{\frac{5}{2}}}\\ &+\frac{144\sqrt{2}(a^2+b^2)^2}{((k-2\pi b)^2+4\pi^2a^2)^5}+\frac{144\sqrt{2}(a^2+b^2)^2}{((k+2\pi b)^2+4\pi^2a^2)^5}\Bigg).\end{aligned} \tag{4.75}$$

The estimate of remainder term $\theta_k(b)$ is obtained by substituting (4.65), (4.68), (4.71), (4.74) and (4.75) into (4.61)

$$\begin{aligned}|\theta_k(b)| <&\frac{e^{-2a\pi^2}}{4\pi^2\left(a^2+b^2\right)^2}\\ &\times\Bigg[\frac{12\pi(a^2+b^2)^{\frac{3}{2}}C_k(a,b)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^4\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}\\ &+\frac{12\pi(a^2+b^2)^{\frac{3}{2}}C_k(a,b)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^4\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}\\ &+\frac{12\sqrt{2}\pi(a^2+b^2)^{\frac{3}{2}}C_k(a,b)}{\left((k+2\pi b)^2+4\pi^2a^2\right)^4\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}\\ &+\frac{12\sqrt{2}\pi(a^2+b^2)^{\frac{3}{2}}C_k(a,b)}{\left((k-2\pi b)^2+4\pi^2a^2\right)^4\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{3}{2}}}\\ &+b\left(ak^2-(a^2+b^2)\right)\\ &\times\Bigg(\frac{144\sqrt{2}(a^2+b^2)^2}{\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}}+\frac{144\sqrt{2}(a^2+b^2)^2}{\left((k+2\pi b)^2+4\pi^2a^2\right)^5}\\ &+\frac{144\sqrt{2}(a^2+b^2)^2}{\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}}+\frac{144\sqrt{2}(a^2+b^2)^2}{\left((k-2\pi b)^2+4\pi^2a^2\right)^5}\Bigg)\Bigg].\end{aligned}$$

By taking

$$\frac{12(a^2+b^2)^{\frac{3}{2}}}{\left((k+2\pi b)^2+4\pi^2a^2\right)^5\left((k-2\pi b)^2+4\pi^2a^2\right)^5}$$

common from all expressions, we write

$$\begin{aligned}|\theta_k(b)| <&\frac{3e^{-2a\pi^2}}{\pi^2\sqrt{a^2+b^2}\left((k+2\pi b)^2+4\pi^2a^2\right)^5\left((k-2\pi b)^2+4\pi^2a^2\right)^5}\\ &\times\Bigg[\pi C_k(a,b)\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right)\\ &\times\Bigg(\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}+\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\\ &+\sqrt{2}\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}+\sqrt{2}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\Bigg)\\ &+12\sqrt{2}b\sqrt{a^2+b^2}\left(ak^2-(a^2+b^2)\right)\\ &\times\Bigg(\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}+\left((k-2\pi b)^2+4\pi^2a^2\right)^5\\ &+\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}+\left((k+2\pi b)^2+4\pi^2a^2\right)^5\Bigg)\Bigg].\end{aligned}$$

Substituting the value of $C_k(b)$ from (4.64)

$$\begin{aligned}
|\theta_k(b)| <& \frac{3\mathrm{e}^{-2a\pi^2}}{\pi^2\sqrt{a^2+b^2}\left((k+2\pi b)^2+4\pi^2a^2\right)^5\left((k-2\pi b)^2+4\pi^2a^2\right)^5} \\
&\times\Bigg[\pi\Bigg(\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right) \\
&\times\left(\left((a^2-b^2)k^2+4(a^2+b^2)\pi^2\right)^2+4b^2\left(ak^2-2(a^2+b^2)\right)^2\right)^{\frac{1}{2}} \\
&+8\pi b\left(a^2+b^2\right)\left(4ak^2-(4a^2+b^2)\right)\left(\left(3k^2-4(a^2-b^2)\pi^2\right)^2+64a^2b^2\pi^4\right)^{\frac{1}{2}}\Bigg) \\
&\times\left((k+2\pi b)^2+4\pi^2a^2\right)\left((k-2\pi b)^2+4\pi^2a^2\right) \\
&\times\Bigg(\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}+\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}} \\
&+\sqrt{2}\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}+\sqrt{2}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\Bigg) \\
&+12\sqrt{2}b\sqrt{a^2+b^2}\left(ak^2-(a^2+b^2)\right) \\
&\times\Bigg(\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}} \\
&+\left((k-2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}}\left((k+2\pi b)^2+4\pi^2a^2\right)^{\frac{5}{2}} \\
&+\left((k-2\pi b)^2+4\pi^2a^2\right)^5+\left((k+2\pi b)^2+4\pi^2a^2\right)^5\Bigg)\Bigg].
\end{aligned}$$

We write $|\theta_k(b)|$ in order notation

$$\begin{aligned}
|\theta_k(b)| &= \frac{3\mathrm{e}^{-2a\pi^2}}{\pi^2\sqrt{a^2+b^2}}2\pi(a^2+b^2)(1+\sqrt{2})\mathcal{O}(k^{-5}) \\
&= \frac{6(1+\sqrt{2})\sqrt{a^2+b^2}\mathrm{e}^{-2a\pi^2}}{\pi}\mathcal{O}(k^{-5}). \qquad (4.76)
\end{aligned}$$

From (4.54) and (4.76), we write

$$\left|\frac{G'_k(b)}{\widetilde{G'}_k(b)}-1\right| = \left|\frac{\theta_k(b)}{\widetilde{G'}_k(b)}\right| = \frac{|\theta_k(b)|}{\left|\widetilde{G'}_k(b)\right|} = \frac{3\pi(1+\sqrt{2})\sqrt{a^2+b^2}}{4b}\mathcal{O}\left(k^{-1}\right).$$

Therefore, $G'_k(b)\sim\widetilde{G'}_k(b)$ for all $(a,b)\in\Omega_R$ when $|k|>N_0(a,b)$. The approximation term $\widetilde{G'}_k(b)>0$ and the remainder term has the faster decay than the approximation term. Hence, $G'_k(b)>0$ for all $(a,b)\in\Omega_R$ when $|k|>N_0(a,b)$.

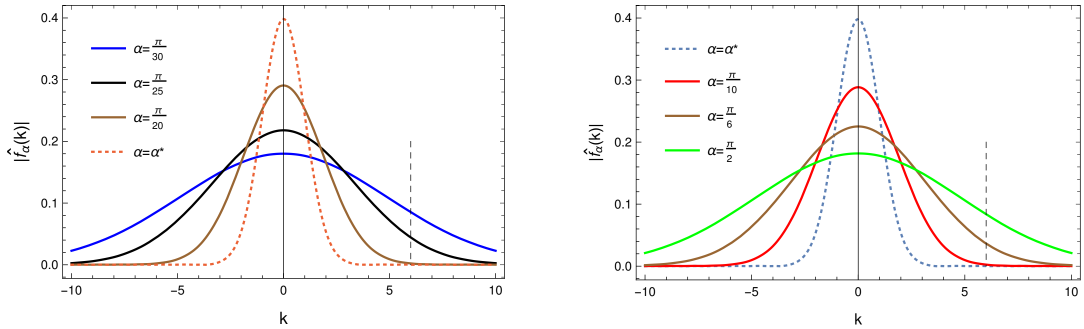
The domain Ω_L

We shall prove $G_k'(b) < 0$ for all $(a, b) \in \Omega_L$ when $|k| > N_0(a, b)$. From (4.51), we find $G_k'(-b)$ for all $k \in \mathbb{Z}$ by replacing b by $-b$ as

$$\begin{aligned}
G_k'(-b) =& \frac{\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{32\pi^{\frac{3}{2}}(a^2+b^2)^{\frac{5}{2}}} \\
&\times \Bigg[-2\sqrt{\pi} b \left(ak^2 - (a^2+b^2)\right) \left| \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b) - 2\pi a}{2\sqrt{a - \mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b) + 2\pi a}{2\sqrt{a-\mathrm{i}b}} \right) \right|^2 \\
&+ \sqrt{a+\mathrm{i}b}(a-\mathrm{i}b)^2 \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b) - 2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b) + 2\pi a}{2\sqrt{a-\mathrm{i}b}} \right) \\
&\times \left(((k+2\pi b) - \mathrm{i}2\pi a)\, \mathrm{e}^{\frac{-(\mathrm{i}(k-2\pi b)-2\pi a)^2}{4(a+\mathrm{i}b)}} - ((k-2\pi b) + \mathrm{i}2\pi a)\, \mathrm{e}^{\frac{-(\mathrm{i}(k+2\pi b)+2\pi a)^2}{4(a+\mathrm{i}b)}} \right) \\
&+ \sqrt{a-\mathrm{i}b}(a+\mathrm{i}b)^2 \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \\
&\times \left(((k+2\pi b) + \mathrm{i}2\pi a)\, \mathrm{e}^{\frac{-(\mathrm{i}(k-2\pi b)+2\pi a)^2}{4(a-\mathrm{i}b)}} - ((k-2\pi b) - \mathrm{i}2\pi a)\, \mathrm{e}^{\frac{-(\mathrm{i}(k+2\pi b)-2\pi a)^2}{4(a-\mathrm{i}b)}} \right) \Bigg] \\
=& \frac{-\mathrm{e}^{\frac{-ak^2}{2(a^2+b^2)}}}{32\pi^{\frac{3}{2}}(a^2+b^2)^{\frac{5}{2}}} \\
&\times \Bigg[2\sqrt{\pi} b \left(ak^2 - (a^2+b^2)\right) \left| \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a + \mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \right|^2 \\
&+ \sqrt{a-\mathrm{i}b}(a+\mathrm{i}b)^2 \operatorname{erf}\left(\frac{\mathrm{i}(k-2\pi b) - 2\pi a}{2\sqrt{a+\mathrm{i}b}}, \frac{\mathrm{i}(k+2\pi b) + 2\pi a}{2\sqrt{a+\mathrm{i}b}} \right) \\
&\times \left(((k-2\pi b) - \mathrm{i}2\pi a)\, \mathrm{e}^{\frac{-(\mathrm{i}(k+2\pi b)-2\pi a)^2}{4(a-\mathrm{i}b)}} - ((k+2\pi b) + \mathrm{i}2\pi a)\, \mathrm{e}^{\frac{-(\mathrm{i}(k-2\pi b)+2\pi a)^2}{4(a-\mathrm{i}b)}} \right) \\
&+ \sqrt{a+\mathrm{i}b}(a-\mathrm{i}b)^2 \operatorname{erf}\left(\frac{\mathrm{i}(k+2\pi b) - 2\pi a}{2\sqrt{a-\mathrm{i}b}}, \frac{\mathrm{i}(k-2\pi b) + 2\pi a}{2\sqrt{a-\mathrm{i}b}} \right) \\
&\times \left(((k-2\pi b) + \mathrm{i}2\pi a)\, \mathrm{e}^{\frac{-(\mathrm{i}(k+2\pi b)+2\pi a)^2}{4(a+\mathrm{i}b)}} - ((k+2\pi b) - \mathrm{i}2\pi a)\, \mathrm{e}^{\frac{-(\mathrm{i}(k-2\pi b)-2\pi a)^2}{4(a+\mathrm{i}b)}} \right) \Bigg] \\
=& -G_k'(b).
\end{aligned}$$

Since $G_k'(b) > 0$ for all $(a, b) \in \Omega_R$ when $|k| > N_0(a, b)$, we have $G_k'(-b) < 0$ for all $(a, b) \in \Omega_L$ when $|k| > N_0(a, b)$. ■

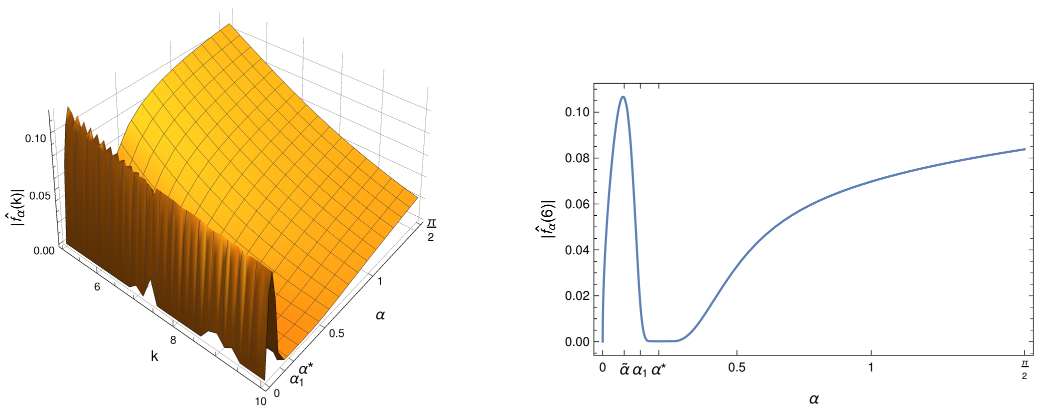
We have proved that decay in fractional Fourier coefficients of large degree follows monotone behavior. The fractional Fourier coefficients $\left|\hat{f}_\alpha(k)\right|$ for various fractional Fourier parameters are plotted in Figure 4.9.



(a) (b)

Figure 4.9: For the chirp $f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}$ with $\gamma = \frac{1}{2\pi}$ and $\mu = \frac{3}{4}$, the graph of $\left|\hat{f}_\alpha(k)\right|$ over $-10 \leq k \leq 10$ (a) for some α on the domain $0 < \alpha \leq \alpha^* \approx 0.209$, and (b) for some some α on the domain $0.209 \approx \alpha^* \leq \alpha \leq \frac{\pi}{2}$.

In Figure 4.2, we observe that by choosing $\gamma = \frac{1}{10\pi}$ and $\mu = \frac{3}{4}$ the fractional Fourier coefficients $\left|\hat{f}_\alpha(k)\right|$ are not monotone with respect to α whereas the Figure 4.9 illustrates that by choosing $\gamma = \frac{1}{2\pi}$ and $\mu = \frac{3}{4}$ with $(\gamma, \mu) \in \mathcal{D}$, the fractional Fourier coefficients are monotone. The three-dimensional plot of the fractional Fourier coefficients of exponential linear chirps $\left|\hat{f}_\alpha(k)\right|$ and plot for $\left|\hat{f}_\alpha(6)\right|$ on fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$ is shown in Figure 4.10.



(a) (b)

Figure 4.10: For the chirp $f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}$ with $\gamma = \frac{1}{2\pi}$ and $\mu = \frac{3}{4}$, (a) the three-dimensional plot of $\left|\hat{f}_\alpha(k)\right|$ for $5 \leq k \leq 10$, and (b) the graph of $\left|\hat{f}_\alpha(6)\right|$ on fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$.

4.3.3 The L^2 Error by Fractional Fourier Series Approximation

The L^2 error in approximation of exponential linear chirps by partial sums of FrFS follows the monotonicity property which is a consequence of Theorem 4.3.2.

Corollary. *Let*

$$f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi]$$

be an exponential linear chirp defined for given $\gamma > 0$ *and* $\mu \in \mathbb{R}^+$. *Let* $(\gamma, \mu) \in \mathcal{D}$ *given in* (4.28). *Then there exists* $N \in \mathbb{N}_0$ *and* $\alpha_1 = \alpha_1(\gamma, \mu)$ *such that for all* $N > N(\gamma, \mu)$, *the* L^2 *error on* $[-\pi, \pi]$ *is monotone. i.e.,*

$$\left\| f_\gamma^\mu - S_{N,\alpha_R} \right\|_{L^2[-\pi,\pi]} < \left\| f_\gamma^\mu - S_{N,\beta_R} \right\|_{L^2[-\pi,\pi]}$$

for $\alpha^* < \alpha_R < \beta_R \leq \frac{\pi}{2}$ *and*

$$\left\| f_\gamma^\mu - S_{N,\beta_L} \right\|_{L^2[-\pi,\pi]} > \left\| f_\gamma^\mu - S_{N,\alpha_L} \right\|_{L^2[-\pi,\pi]},$$

for $\alpha_1 \leq \beta_L < \alpha_L < \alpha^*$ *with* $0 < \alpha_1 < \alpha^*$, *where the parameter* α^* *is the optimal fractional Fourier parameter, and*

$$S_{N,\alpha}(t) = \sum_{|k| \leq N} \hat{f}_\alpha(k) \phi_{k,\alpha}(t), \quad N \in \mathbb{N}_0$$

is the N*-th partial sum for the fractional Fourier series.*

Proof. Let

$$J_N(\alpha) = \left\| f_\gamma^\mu - S_{N,\alpha} \right\|_{L^2[-\pi,\pi]}^2.$$

Using Parseval's identity, $J_N(\alpha)$ becomes

$$J_N(\alpha) = \sum_{|k|>N} \left| \hat{f}_\alpha(k) \right|^2 = \sum_{|k|>N} F_k(\alpha).$$

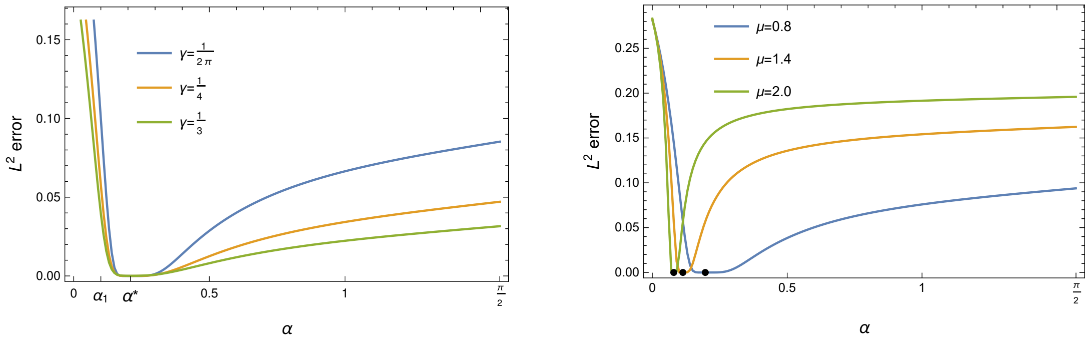
The derivative of $J_N(\alpha)$ with respect to α, denoted by $J_N'(\alpha)$, is given by

$$J_N'(\alpha) = \sum_{|k|>N} F_k'(\alpha).$$

In Theorem 4.3.2, we have proved $F_k'(\alpha) > 0$ for all $\alpha^* < \alpha \leq \frac{\pi}{2}$ when $|k| > N(\gamma, \mu)$. Therefore, $J_N'(\alpha) > 0$ on $\alpha^* < \alpha \leq \frac{\pi}{2}$ when $|k| > N(\gamma, \mu)$.

Again from Theorem 4.3.2, we have $F_k'(\alpha) < 0$ for all $\alpha_1 \leq \alpha < \alpha^*$ with $0 < \alpha_1 < \alpha^*$ when $|k| > N(\gamma, \mu)$. Therefore, $J_N'(\alpha) < 0$ on $\alpha_1 \leq \alpha < \alpha^*$ when $|k| > N(\gamma, \mu)$. ■

The monotonicity in L^2 error is illustrated in Figure 4.11.



(a) (b)

Figure 4.11: For the chirp $f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}$, the L^2 error in FrFS approximation of exponential linear chirps by using $N = 3$ with (a) $\mu = \frac{3}{4}$ and (b) $\gamma = \frac{1}{2\pi}$.

4.4 Conclusion and Outlook

The main intent of this thesis was to find the approximation of exponential linear chirps by the partial sums of FrFS on a finite interval $[-\pi, \pi]$. Such an approach incorporates an optimal parameter technique for which the FrFS possesses an advantage over the classical Fourier series. To sum up, the objectives of this thesis were

i) to approximate the exponential linear chirps of the form

$$f(t) = f_\gamma^\mu(t) = \mathrm{e}^{-\pi(\gamma+\mathrm{i}\mu)t^2}, \quad t \in [-\pi, \pi],$$

by the partial sums of FrFS which uses fractional Fourier basis functions

$$\left\{\phi_{k,\alpha}(t) = \mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\cot\alpha}\mathrm{e}^{\mathrm{i}kt} : k \in \mathbb{Z}\right\} \quad \text{for} \quad 0 < \alpha \leq \frac{\pi}{2},$$

ii) to prove the inequalities of fractional Fourier coefficients of exponential linear chirps on the fractional Fourier parameter domain $0 < \alpha \leq \frac{\pi}{2}$

iii) and to prove that the FrFS provides the best results when fractional parameter

$$\alpha^* = \arctan\left(\frac{1}{2\pi\mu}\right)$$

is used.

In Chapter 4, we have proved optimality properties of fractional Fourier coefficients for all $\gamma > 0$ and $\mu \in \mathbb{R}^+$, and the monotonicity properties on a domain $\mathcal{D}$, where

$$\mathcal{D} = \left\{(\gamma, \mu) : \gamma \geq \frac{1}{2\pi},\ 0 < \mu \leq \frac{\sqrt{\gamma}}{8\pi}\left(\lambda\mathrm{e}^{\pi^3\gamma} + \sqrt{\left(\lambda\mathrm{e}^{\pi^3\gamma}\right)^2 - 64\pi^2\gamma}\right)\right\}$$

with

$$\lambda = \left(1 - \frac{2\sqrt{2}\mathrm{e}^{\frac{-\pi^2}{2}}}{\pi^{\frac{3}{2}}}\right)^2.$$

We have studied FrFS from mathematical point of view and proved new inequalities related to fractional Fourier coefficients of exponential linear chirps. The properties of fractional Fourier coefficients of the linear chirps have been investigated by changing the fractional Fourier parameter to different values. The analysis has discovered that as the fractional Fourier parameter α approaches α^*, fractional Fourier coefficients of zero degree are increased to the maximum, and fractional Fourier coefficients of large degree have the fastest decay. The minimum L^2 error in FrFS approximation is achieved when N is large enough. We have proved that the FrFS is a very useful technique for the analysis of chirp functions and it has better performance than the classical Fourier series. Hence, the analysis of exponential linear chirps pursued in this thesis proves to be more widely applicable than the classical Fourier series technique.

Now we want to discuss open questions for further research. It is of interest to determine the smallest value of the parameter α_1 in Theorem 4.3.2. It is proved in this thesis that the fractional Fourier coefficients $\left|\hat{f}_\alpha(0)\right|$ and $\left|\hat{f}_\alpha(k)\right|$ follow monotonicity properties over a domain Ω given by

$$\Omega = \left\{(a, b) : a \geq \frac{1}{2},\ -\Psi(a) \leq b \leq \Psi(a)\right\} = \Omega_L \cup \Omega_R,$$

where

$$\Psi(a) = \frac{\sqrt{a\pi}}{8\pi}\left(\lambda \mathrm{e}^{a\pi^2} + \sqrt{\left(\lambda \mathrm{e}^{a\pi^2}\right)^2 - 64a\pi}\right)$$

and

$$\lambda = \left(1 - \frac{2\sqrt{2}\mathrm{e}^{\frac{-\pi^2}{2}}}{\pi^{\frac{3}{2}}}\right)^2.$$

The monotonicity domain Ω can be enlarged by finding the sharp estimates of the integrals

$$\int_0^1 \mathrm{e}^{-a\pi^2 u^2} \cos\left(b\pi^2 u^2\right)\, \mathrm{d}u \quad \text{and} \quad \int_0^1 \mathrm{e}^{-a\pi^2 u^2} \sin\left(b\pi^2 u^2\right)\, \mathrm{d}u,$$

given in Lemma 3.4.2.

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List of Figures

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