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Weak Convergence of the Milstein Scheme for Semi-Linear Parabolic Stochastic Evolution Equations

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Zusammenfassung

Die Numerik des Milstein-Verfahrens für stochastische gewöhnliche Differenzialgleichungen (SDEs) wurde bereits umfassend untersucht. Es konvergiert sowohl mit starker als auch schwacher Ordnung eins. Im Gegensatz dazu ist über das Milstein-Verfahren für stochastische *partielle* Differenzialgleichungen oder allgemeinere stochastische Evolutionsgleichungen (SEEs) noch wenig bekannt.

Diese Arbeit beschäftigt sich mit der schwachen Konvergenz des Milstein-Verfahrens im Kontext von SEEs. Wir beweisen, dass es, ähnlich zum SDE-Fall, von der Ordnung fast eins ist – genauer, von der Ordnung $1 - \varepsilon$ für alle $\varepsilon > 0$. Wir bedienen uns dabei der Halbgruppentheorie von Da Prato und Zabczyk und untersuchen die Approximation sogenannter milder Lösungen von Gleichungen des semilinearen parabolischen Typs. Insbesondere lassen wir zu, dass der Driftkoeffizient der Evolutionsgleichung Werte in gewissen, mit dem dominierenden linearen Operator assoziierten, Distributionsräumen annimmt. In diesem Fall hängt die Konvergenzordnung von der Regularität der Koeffizienten ab und strebt für abnehmende Regularität gegen null.

Der Beweis verwendet Elemente der milden stochastischen Analysis, welche vor Kurzem von Da Prato, Jentzen und Röckner (*Trans. Amer. Math. Soc.*, 372(6), 2019) eingeführt wurde, und hängt entscheidend von neuen Resultaten über die Regularität der Lösungen der assoziierten unendlich-dimensionalen Kolmogorow-Rückwärtsgleichung von Andersson, Hefter, Jentzen und Kurniawan (*Potential Anal.*, 50(3), 2019) ab. Er basiert auf der Arbeit von Jentzen und Kurniawan (*Found. Comput. Math.*, 21(2), 2021), die Verfahren vom Euler-Typ untersucht.

Abstract

The numerical analysis of the Milstein scheme for stochastic ordinary differential equations (SDEs) is relatively well understood. It converges with both strong and weak order one. However, much less is known about the Milstein scheme and its variants when applied to stochastic *partial* differential equations or more general stochastic evolution equations.

This thesis focuses on the weak convergence of the Milstein scheme in the latter setting. We prove that, similar to the SDE case, it also achieves an order of almost one — specifically, an order of $1-\varepsilon$ for all $\varepsilon > 0$. More concretely, we work in the semigroup framework introduced by Da Prato and Zabczyk and examine the approximation of mild solutions of equations of semi-linear parabolic type. In addition, we allow the drift coefficient of the evolution equation to take values in certain distribution spaces associated to the dominating linear operator. In that case, the order of convergence depends on the regularity of the coefficients and tends to zero as the regularity decreases.

The proof employs elements of the mild stochastic calculus recently introduced by Da Prato, Jentzen and Röckner (*Trans. Amer. Math. Soc.*, 372(6), 2019) and crucially depends on recent results on the regularity of solutions to the associated infinite-dimensional Kolmogorov backward equation by Andersson, Hefter, Jentzen and Kurniawan (*Potential Anal.*, 50(3), 2019). It is based on work by Jentzen and Kurniawan investigating Euler-type schemes (*Found. Comput. Math.*, 21(2), 2021).

Contents

List of Symbols

Notation	Description
$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$	extended real numbers
$a \wedge b := \min\{a, b\}$	minimum of $a, b \in \mathbb{R}$
$\lfloor t \rfloor_h := \lfloor t/h \rfloor \cdot h$	round down to nearest multiple of h
H, U, V	Hilbert spaces
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$	filtered probability space
$\mathbb{M}(\mathcal{A}, \mathcal{B})$	space of $\mathcal{A}$-$\mathcal{B}$-measurable functions
$L^0(\Omega, H) := \mathbb{M}(\mathcal{F}, \mathcal{B}(H))$	space of random variables with the topology induced by convergence in probability
$L^p(\Omega, H) = L^p(\Omega, \mathcal{F}, \mathrm{P}; H, \mathcal{B}(H))$	Bochner–Lebesgue spaces, $p > 0$
$L^{(k)}(U_1 \times \cdots \times U_k, V)$	space of continuous k-linear functions
$\mathrm{L}_2(U, V)$	space of Hilbert–Schmidt operators
$\mathrm{Lip}^k(U, V)$	space of k-times differentiable functions with Lipschitz derivatives
$\lvert F \rvert_{\mathrm{Lip}(U,V)}$	Lipschitz constant of F
$\tau_p = \begin{cases} \left(1 + \frac{2}{2-p}\right)^{\frac{1}{p}} & p \in (0,2) \\ \sqrt{\frac{p(p-1)}{2}} \cdot \left(\frac{p}{p-1}\right)^{\frac{p}{2}} & p \geq 2 \end{cases}$	constant in the Burkholder–Davis–Gundy inequality
$\kappa_p = \begin{cases} 1 & p = 0 \\ 2^{\frac{1}{p}-1} & p \in (0,1) \\ 1 & p \geq 1 \end{cases}$	constant in the modified triangle inequality
$\Lambda := \{(s,t) \in [0,T]^2 \mid s < t\}$	triangular domain of an evolution family
$\mathrm{D}(A)$	domain of a linear operator A
$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda - A \colon \mathrm{D}(A) \to H \text{ is bijective}\}$	resolvent set of $(A, \mathrm{D}(A))$
$R(\lambda, A) := (\lambda - A)^{-1} \in L(H, H)$	resolvent of $(A, \mathrm{D}(A))$ for $\lambda \in \rho(A)$
$\Sigma_\delta := \{\lambda \in \mathbb{C} \mid \lvert\arg \lambda\rvert < \delta\} \setminus \{0\}$	open sector in the complex plane
$\mathrm{E}_\alpha(x)$	Mittag-Leffler function, $\alpha \in (0, \infty)$

Introduction

As we are trying to model the world in ever greater detail, some are incorporating hundreds of experimental relationships into their models, others are treating these higher order effects as statistical noise. This latter approach leads to the study of partial differential equations, where some parameters are considered as stochastic quantities — so-called *stochastic partial differential equations* (SPDEs).

The use of SPDEs in the sciences has increased steadily in recent years. However, only in trivial cases is it possible to derive an exact formula for the solution to an SPDE. This implies the need to numerically simulate approximations of these solutions. The high dimensionality inherent in infinite-dimensional state-spaces of deterministic PDEs combined with also infinite-dimensional noise makes this an expensive endeavour. It is therefore important to study and develop efficient algorithms that respect the limited time and computational resources of researchers. In the case of stochastic ordinary differential equations (SDEs) there already exists a wide array of different algorithms ranging from general purpose to highly specialised (see [KP92]). For SPDEs on the other hand, there are still open questions even about simple schemes.

After the proposal of any new algorithm it is necessary to assess its correctness and plausibility. In general, an algorithm should converge in some sense to the true solution of the problem at hand as parameters controlling the algorithm's precision are varied.

The most commonly studied kinds of convergence for SPDEs are the so-called strong and weak convergence. Strong convergence refers to the idea that for each realisation of the driving stochastic process, the sample path of the approximation should be close to that of the true solution. It is usually measured in the $L^1(\Omega)$-norm or a stronger $L^p(\Omega)$-norm. In contrast, weak convergence is only concerned with an aggregate convergence over all sample paths and is measured using a class of test functions. In real-world applications we cannot know the specific realisation of the noise and are normally interested only in expected outcomes or most likely scenarios. Thus, focusing on weak convergence is in some cases more appropriate.

In the case of SPDEs we usually have to discretise time, the spatial domain and the noise. In this thesis, we focus on the discretisation in time and thus we are concerned with convergence, in particular weak convergence, as the number of time steps increases or equivalently as the size of the steps decreases.

The main objects of study are *semi-linear parabolic stochastic evolution equations* (SEEs), that means equations on a separable Hilbert space H of the form

$$\begin{cases} \mathrm{d}X_t = [AX_t + F(X_t)]\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t, & t \in (0,T], \\ X_0 = \xi \end{cases}$$

with a closed linear operator $A\colon \mathrm{D}(A) \subset H \to H$, possibly non-linear drift and diffusion terms F and B and a cylindrical Wiener process W on a second Hilbert space U. Parabolic in this context means, that the operator A is assumed to generate an analytic semigroup $(e^{tA})_{t\geq 0}$ (see e.g. [Ama95, Section II.1.2]). The equation is called semi-linear because the non-linear terms are assumed to be of lower order compared to the dominating linear operator. Specifically, this means that F can take values in $H_{-\alpha}$ for $\alpha \in [0,1)$, where $(H_r)_{r\in\mathbb{R}}$ is a family of interpolation (and extrapolation) spaces associated to A (see section 1.2.1 for the precise definition) and taking $\alpha = 1$ would be “the same order” as A. Similarly, the diffusion coefficient will be assumed to take values in the space of Hilbert–Schmidt operators $\mathrm{L}_2(U, H_{-\beta})$ for $\beta \in [0, 1/2)$. The parameters α and β can be thought of as describing the regularity of the coefficient functions, where, at least heuristically, the low Hölder regularity of the Wiener process forces β to be less than 1/2. Moreover, we assume that the non-linearities satisfy a Lipschitz condition.

Under these assumptions a general result (see Theorem 1.28) guarantees the existence of a unique mild solution given by a stochastic process $X\colon \Omega \times [0,T] \to H$ with $X_0 = \xi$ and

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A} B(X_s)\,\mathrm{d}W_s, \quad t \in (0,T].$$

The time-discrete approximation of this mild solution using a class of Milstein-type schemes is the main topic of this thesis. Most notably, this class includes the exponential and the linear-implicit Milstein schemes. As an introductory example, we present the exponential Milstein scheme introduced in [JR15]. It is defined as a time-discrete stochastic process Z^N on an equidistant grid $\{t_0 = 0, \dots, t_N = T\}$ with $N+1$ time points by

$$\begin{aligned} Z^N_0 &= \xi \\ Z^N_{t_{n+1}} &= e^{hA}\Big[Z^N_{t_n} + F(Z^N_{t_n})\cdot h + \int_{t_n}^{t_{n+1}} B(Z^N_{t_n})\,\mathrm{d}W_s \\ &\qquad + \int_{t_n}^{t_{n+1}}\int_{t_n}^{r} B'(Z^N_{t_n})B(Z^N_{t_n})\,\mathrm{d}W_u\,\mathrm{d}W_r\Big], \quad n \in \{0,\dots,N-1\}. \end{aligned}$$

The defining feature of Milstein-type schemes is the presence of a term involving twice iterated stochastic integrals. The approximation (and also the simulation) of these iterated integrals is a difficult topic even in the finite-dimensional case. See [KR23] for a review of a class of methods targeting the $L^2(\Omega)$-approximation in the finite-dimensional case. For the approximation in the infinite-dimensional case we refer to [LR19]. Unfortunately, for the integrand in the iterated integral to be well-defined, we can only allow $\beta = 0$ even though the mild solution exists even for $\beta \in [0, 1/2)$.

For an overview of different Milstein-type schemes we refer to [DJR19, Section 3.3.2]. The strong convergence of these algorithms was studied for example in [LCP10; LCP11] for certain Zakai-type equations, in [JR15] for equations driven by a Wiener process and in [BL12; BL13] for equations driven by general martingales. Furthermore, derivative-free variants of the scheme from [JR15] have been proposed in [HR23; LR18; WG13a]. All of these approximations are shown to converge with a strong order of almost 1 under suitable regularity assumptions as well as only allowing trace-class noise.

The only work dealing with non-linear drift and diffusion coefficients is [JR15], requiring only twice Fréchet differentiability and some specific boundedness conditions on the derivatives. Additionally, the coefficients are allowed to be mappings $F\colon H_\beta \to H$ (resp. $B\colon H_\beta \to \mathrm{L}_2(U_0, H)$) for $\beta \in [0,1)$. While the required boundedness conditions imply that the coefficients can be continuously extended to globally Lipschitz mappings $F\colon H \to H$ (resp. $B\colon H \to \mathrm{L}_2(U_0, H)$), the extensions only have to be differentiable on a dense subspace.

The study of the weak approximation of solutions to SPDEs began relatively recently in the early 2000s. Most proofs are similar to weak convergence proofs for SDEs, usually trying to exploit the regularity properties of the solution to the associated Kolmogorov backward equation. These regularity properties have been studied for additive noise for example in [Cer01; DZ02; RS06] and more recently for more general noise in [And+19; BD18; Deb11]. The weak convergence of fully discrete schemes for equations with additive noise was investigated in [AKL15; BCW22; DP09; Hau03; KLL13; KLP20; Sha03] and for multiplicative noise in [TN16]. Spatially semi-discrete schemes for additive noise were explored in [GKL09; KLL12] and in [AL16; CJK19; JJW21] also for multiplicative noise. The weak convergence of temporally semi-discrete schemes for equations with additive noise was studied in [BG19; Wan16; WG13a] and for multiplicative noise in [dD06; HJK16; JK21]. An approach only utilising the Malliavin calculus was presented in [Kru14]. However, again only the additive noise case is covered.

All of these works are concerned with Euler-type schemes for the time-discretisation in the broader sense that they do not involve any iterated stochastic integrals. Usually, a convergence rate equivalent to $1 - \max\{\alpha, 2\beta\}$ in our setting is proven. That is, order one for regular coefficients with a decrease in the order proportional to the (ir)regularity of the coefficients.

To the best of our knowledge, no one studied the weak convergence of a Milstein-type scheme yet. This could be due to our intuitions based on the finite-dimensional case, that we do not expect the weak convergence rate to exceed the strong convergence rate. But in that case the weak rate can immediately be inferred from the strong rate using the Lipschitz property of the test functions. We provide three reasons why we nevertheless pursued this topic. First, we cannot expect all of our finite-dimensional intuitions to hold in the infinite-dimensional setting. It could be possible that a Milstein scheme for SPDEs attains an even higher order. Ideally, this would be complemented by a proof that we, in fact, cannot exceed an order of $1 - \alpha$. Second, in the generality of our setting no corresponding strong convergence result is currently known and the weak convergence can therefore not be deduced. Third, it is interesting in itself to study whether the used arguments can be adapted to work on Milstein-type schemes.

Let us now present an example of our main result. We additionally require that the non-linear terms are in $\mathrm{Lip}^4(H, H_{-\alpha})$ (respectively $\mathrm{Lip}^4(H, \mathrm{L}_2(U, H))$), i.e. they have to be four times Fréchet differentiable and together with their derivatives satisfy a Lipschitz condition. Fix a separable Hilbert space V. As test functions we allow V-valued functions, that also have to be in the class $\mathrm{Lip}^4(H, V)$. The main results of this thesis (Theorems 3.10 and 3.11) then imply the following weak convergence rates for the exponential Milstein scheme.

Theorem.
Let $\xi \in L^5(\Omega, H)$ and $\varphi \in \mathrm{Lip}^4(H, V)$. If $\alpha \in [0, 1/2)$, then for all $\varepsilon > 0$ there exists a constant $C \in [0, \infty)$ such that for all $N \in \mathbb{N}$ we have

$$\left\| \mathbb{E}\Big[\varphi(Z_T^N) - \varphi(X_T)\Big] \right\|_V \leq C \cdot N^{-(1-\alpha-\varepsilon)}.$$

If $\alpha \in [1/2, 1)$, then for all $\varepsilon > 0$ there exists a constant $C \in [0, \infty)$ such that for all $N \in \mathbb{N}$ we have

$$\left\| \mathbb{E}\Big[\varphi(Z_T^N) - \varphi(X_T)\Big] \right\|_V \leq C \cdot N^{-(\frac{1-\alpha}{4\alpha-1}-\varepsilon)}.$$

We will say that a numerical scheme has an order of almost γ if it has an order of $\gamma - \varepsilon$ for arbitrarily small $\varepsilon > 0$.

The curious drop in the order of convergence for more irregular drift terms ($\alpha \in [1/2, 1)$) results from the used regularisation strategy. In this regime, we have to sacrifice a fraction of our convergence order to counteract singularities that arise when we remove the regularisation. Relative to the expected optimal order of almost $1 - \alpha$ we suffer a reduction of up to two thirds asymptotically for $\alpha \to 1$. This can probably be avoided, if better regularity estimates for the solution of the Kolmogorov backward equation become available. In particular, we suspect that the results in [And+19], on which our proof is based, can be strengthened under the additional assumption that $\beta = 0$.

Compared to other weak convergence results in the literature the class of test functions considered here is slightly smaller, as usually test functions in $C_b^2(H, \mathbb{R})$ or $C_b^3(H, \mathbb{R})$ are considered. However, the choice of test functions (as well as the required differentiability of the drift and diffusion coefficients) is essentially dictated by the regularity result of the Kolmogorov backward equation from [And+19] that we employ.

Let us compare this result to the strong convergence result in [JR15]. Although the setting is different we can in some cases translate between them. Specifically, if we limit our regularity parameter to $\alpha \in (0, 1/2)$ and choose β and δ in the setting of [JR15] equal to α we can reconcile the two settings by considering their setting applied to the modified equation given by

$$\begin{aligned} \tilde{F}&\colon H_\alpha \to H, \quad x \mapsto A^{-\alpha} F(A^\alpha x), \\ \tilde{B}&\colon H_\alpha \to \mathrm{L}_2(U, H_\alpha), \quad x \mapsto A^{-\alpha} B(A^\alpha x) \end{aligned}$$

and using the initial value $\tilde{\xi} = A^{-\alpha}\xi$. To guarantee a strictly positive strong order we additionally have to require that the initial value is smooth enough, specifically $\xi \in L^5(\Omega, H_\eta)$ for some $\eta \in (0, 1/2)$ so that $\tilde{\xi} \in L^5(\Omega, H_{\eta+\alpha})$. The result in [JR15] then yields a strong order of convergence of $\min(2\eta, \eta + \alpha)$. Choosing a regular initial value $\xi \in L^5(\Omega, H_{\frac{1}{2}})$ allows to take η arbitrarily close to $1/2$, giving a strong order of almost $1/2 + \alpha$. Compare this to the weak order of almost $1 - \alpha$ that we get even for $\eta = 0$. Interestingly, the strong order seems to improve for more irregular drift functions (i.e. increasing α) whereas the weak order gets worse. In particular, the weak order is higher than the strong order in the small α regime. This is in stark contrast to the finite-dimensional theory where the classical Milstein scheme has the same strong and weak order, both equal to 1. A reason could be that, unlike in our case, their proof depends on the regularity of the initial value. Indeed, if $\eta = 0$ the result in [JR15] does not yield convergence at all.

We use and adapt the arguments given in [JK21] for the weak convergence of Euler-type schemes. A similar approach was also used in [CJK19] to prove weak convergence rates of spectral Galerkin approximations. Additionally, the preprint [HJK16] generalises the results from [JK21] to a Banach space setting. The main tools used are the mild Itô formula introduced in [DJR19] and the regularity estimates for the Kolmogorov backward equation associated to the true solution presented in [And+19].

Let us now provide a short outline of the proof. First, we regularise (or mollify) the initial value as well as the coefficients of the equation separately. This is done using the semigroup generated by the dominating linear operator, which is assumed to be analytic, and thus provides a strong regularisation effect. To estimate the weak error for this regularised equation we insert an auxiliary process and employ the triangle inequality. After rewriting the error in terms of the solution of the Kolmogorov backward equation associated to the true solution evaluated at this auxiliary process, we then show that the auxiliary process is a suitably regular strong solution of an SPDE to allow the application of the strong Itô formula. The resulting terms are then estimated using a combination of the mild Itô formula, the regularity properties of the solution of the Kolmogorov equation and classical regularity estimates of the analytical semigroup. Here we need to take special care to make the constants explicit and, in particular, to track the dependence on the coefficients of the equation to later ensure that everything stays bounded when we remove the regularisation. Finally, we show how to optimally balance the regularisation parameter with the step size in order to maximise the final rate of convergence.

This thesis consists of three chapters. In chapter 1, we present the necessary background needed for the main parts. In particular, we recall the theory of stochastic integration with respect to a cylindrical Wiener process, the semigroup approach to stochastic evolution equations as well as relevant results from the recent theory of mild stochastic calculus and the needed facts about the Kolmogorov backward equation. We then summarise our approach to the numerical approximation of stochastic evolution equations.

The central chapter 2 represents the main part of the work. After establishing the necessary estimates, we are able to deduce the weak convergence theorem for the regularised equation in Corollary 2.10. The core arguments of this thesis are contained in the proof of Lemma 2.8.

Finally, chapter 3 focuses on the extension of Corollary 2.10 to the general setting. For this, we first recall the useful perturbation estimate Proposition 3.2, which is based on a generalised version of Grönwall's lemma. To motivate this we briefly present a connection to the theory of fractional integrals and fractional integral equations. The perturbation estimate will allow us to carry out the regularisation strategy detailed in section 3.2. Using this regularisation, the main results of this thesis are then presented in section 3.3 as Theorems 3.10 and 3.11.

Chapter 1

Preliminaries

1.1 Stochastic Integration

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let U and H be separable, non-trivial Hilbert spaces and let $(e_n)_{n\in\mathbb{N}} \subset U$ and $(f_n)_{n\in\mathbb{N}} \subset H$ be orthonormal bases of U and H, respectively. Let $T > 0$ be an arbitrary but fixed final time.

Definition 1.1 (Cylindrical Wiener process)**.**
Let $(\beta_n)_{n\in\mathbb{N}}$ be a family of independent, real-valued Wiener processes on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$. Then we call $W = (W_t)_{t\geq 0}$ with

$$W_t = W(t) = \sum_{n\in\mathbb{N}} \beta_n(t) e_n, \quad t \geq 0$$

a cylindrical Wiener process on *U.*

Because $\mathbb{E}[\sum_{n\in\mathbb{N}} |\beta_n(t)|^2] = \sum_{n\in\mathbb{N}} t$ is not finite for $t > 0$, we see that $(W_t)_{t\geq 0}$ does not exist as a square integrable, U-valued stochastic process. However, we can make sense of this object as a family of bounded linear operators $(W_t)_{t\geq 0}$ where

$$\begin{aligned} W \colon [0, \infty) &\to L(U, L^2(\Omega)) \\ t &\mapsto \left[u \mapsto W_t u := \sum_{n\in\mathbb{N}} \beta_n(t) \langle e_n, u\rangle_U \right]. \end{aligned}$$

This is now meaningful as we have $\|W_t u\|^2_{L^2(\Omega)} = t \cdot \|u\|^2_U < \infty$ for every $t \geq 0$ and $u \in U$ by Parseval's identity. Alternatively, it is known that the series in Definition 1.1 converges in any larger Hilbert space U_0 such that the embedding $\iota \colon U \to U_0$ is Hilbert–Schmidt. Then we can regard $(W_t)_{t\geq 0}$ as a standard (nuclear) Q-Wiener process on this larger space U_0 with covariance operator $Q = \iota^*\iota$.

Remark 1.2. The name "cylindrical" comes from the cylindrical measures in the sense of Schwartz [Sch73] which are additive set functions defined on the algebra of cylindrical sets (also called cylinder sets). From that point of view, W is a family of cylindrical random variables $(W_t)_{t\geq 0}$ whose cylindrical distribution is a Gaussian cylindrical measure. See also [Rie11] for a recent exposition of this viewpoint. ◊

1.1.1 L^2 Theory

To construct an integral we must first consider the appropriate integrands. To this end, define the set of *predictable rectangles* as

$$\mathcal{R}_T = \{(s,t] \times F_s \mid 0 \leq s < t \leq T, F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 \mid F_0 \in \mathcal{F}_0\}$$

and the *predictable σ-algebra* on $[0,T] \times \Omega$ as

$$\mathcal{P}_T = \sigma(\mathcal{R}_T)\,.$$

A stochastic process is called *predictable* if it is measurable with respect to the predictable σ-algebra. For two measurable spaces $(\Omega_1, \mathcal{A})$ and $(\Omega_2, \mathcal{B})$ we denote by $\mathbb{M}(\mathcal{A}, \mathcal{B})$ the space of $\mathcal{A}$-$\mathcal{B}$-measurable functions. We will denote by $L^p(\Omega, H) = L^p(\Omega, \mathcal{F}, \mathrm{P}; H, \mathcal{B}(H))$ for $p > 0$ the usual Bochner–Lebesgue spaces and will sometimes use $L^0(\Omega, H) := \mathbb{M}(\mathcal{F}, \mathcal{B}(H))$ equipped with the topology of convergence in probability where it helps to unify the notation of a theorem or definition.

In this work we will need two types of spaces of stochastic processes. For a separable Hilbert space V we define the space of predictable, continuous and p-integrable V-valued processes

$$\mathfrak{C}_T^p(V) = \Big\{X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(V)) \,\Big|\, X \text{ a.s. continuous}, \|X\|_{\mathfrak{C}_T^p(V)} < \infty\Big\}/_{\|\cdot\|_{\mathfrak{C}_T^p(V)}}$$

where $p > 0$ and

$$\|X\|_{\mathfrak{C}_T^p(V)} = \mathbb{E}\bigg[\sup_{t\in[0,T]} \|X_t\|_V^p\bigg]^{\frac{1}{p}}. \tag{1.1}$$

And second the space of predictable, p-q-integrable V-valued processes

$$\mathfrak{L}_T^{p,q}(V) = \Big\{X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(V)) \,\Big|\, \|X\|_{\mathfrak{L}_T^{p,q}(V)} < \infty\Big\}/_{\|\cdot\|_{\mathfrak{L}_T^{p,q}(V)}}$$

where $p > 0$, $q \geq 1$ and

$$\|X\|_{\mathfrak{L}_T^{p,q}(V)} = \mathbb{E}\bigg[\Big(\int_0^T \|X_t\|_V^q \,\mathrm{d}t\Big)^{\frac{p}{q}}\bigg]^{\frac{1}{p}}. \tag{1.2}$$

Furthermore, we define for $p = 0$ the space of almost surely continuous processes

$$\mathfrak{C}_T^0(V) = \{X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(V)) \mid X \text{ a.s. continuous}\}/_{\|\cdot\|_{\mathfrak{C}_T^0(V)}}$$

with

$$\|X\|_{\mathfrak{C}_T^0(V)} = \mathbb{E}\bigg[\frac{\sup_{t\in[0,T]}\|X_t\|_V}{1 + \sup_{t\in[0,T]}\|X_t\|_V}\bigg] \tag{1.3}$$

as well as for $q \geq 1$ the space of almost surely q-integrable functions

$$\mathfrak{L}_T^{0,q}(V) = \Big\{X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(V)) \,\Big|\, \mathrm{P}\Big(\textstyle\int_0^T \|X_t\|_V^q \,\mathrm{d}t < \infty\Big) = 1\Big\}/_{\|\cdot\|_{\mathfrak{L}_T^{0,q}(V)}}$$

with

$$\|X\|_{\mathfrak{L}^{0,q}_T(V)} = \mathbb{E}\left[\frac{\left(\int_0^T \|X_t\|_V^q \,\mathrm{d}t\right)^{\frac{1}{q}}}{1 + \left(\int_0^T \|X_t\|_V^q \,\mathrm{d}t\right)^{\frac{1}{q}}}\right]. \tag{1.4}$$

In all cases we have to take the quotient to identify functions that differ only on $\lambda|_{[0,T]} \otimes \mathrm{P}$-null sets. This ensures that the associated "norms" are positive-definite. This also means that, strictly speaking, all spaces contain equivalence classes of functions. In the following, we will usually identify functions with their corresponding equivalence classes.

All spaces are Banach spaces for $p \geq 1$ and quasi-Banach spaces for $0 < p < 1$. Denoting by E either $\mathfrak{C}^p_T(V)$ or $\mathfrak{L}^{p,q}_T(V)$ for $0 < p < 1$, we have that $d_E(x,y) := \|x - y\|_E^p$ is a metric on E with respect to which it is complete. Additionally, a modified triangle inequality holds as well

$$\|x + y\|_E \leq \kappa_p(\|x\|_E + \|y\|_E) \tag{1.5}$$

where $\kappa_p = 2^{\frac{1}{p}-1}$ for $0 < p < 1$ and we set $\kappa_p = 1$ for $p \geq 1$. For a short overview of quasi-Banach spaces see [Jar81, Chapter 6] or [Köt60, §15.9]. When general statements are true for all $p > 0$ we will sometimes simply refer to norms even though they are only quasinorms for $0 < p < 1$.

In the case $p = 0$, the functions $\|\cdot\|_{\mathfrak{C}^0_T(V)}$ and $\|\cdot\|_{\mathfrak{L}^{0,q}_T(V)}$ define an F-norm, see [Jar81, Section 2.7]. In particular, they satisfy the triangle inequality and we set $\kappa_0 = 1$. The associated metric induces the topology of convergence in probability. Thus, a sequence of functions $(X^n)_{n\in\mathbb{N}} \subset \mathfrak{C}^0_T(V)$ converges to a function $X \in \mathfrak{C}^0_T(V)$, i.e. $\lim_{n\to\infty}\|X^n - X\|_{\mathfrak{C}^0_T(V)} = 0$, if and only if

$$\forall \varepsilon > 0\colon \lim_{n\to\infty} \mathrm{P}\left(\sup_{t\in[0,T]} \|X^n_t - X_t\|_V > \varepsilon\right) = 0$$

and similarly a sequence of functions $(X^n)_{n\in\mathbb{N}} \subset \mathfrak{L}^{0,q}_T(V)$ converges to a function $X \in \mathfrak{L}^{0,q}_T(V)$, if and only if

$$\forall \varepsilon > 0\colon \lim_{n\to\infty} \mathrm{P}\left(\int_0^T \|X^n(t) - X(t)\|_V^q \,\mathrm{d}t > \varepsilon\right) = 0\,.$$

Remark 1.3. All integrals have to be interpreted as Bochner integrals. We follow here the exposition of [Coh13, Appendix E]. Since the Hilbert space V is assumed to be separable, the distinction between Borel measurable and strongly or Bochner measurable functions is not necessary. In particular, the space of Hilbert–Schmidt operators $\mathrm{L}_2(U,H)$ is a separable Hilbert space as U and H are assumed to be separable and $\mathrm{L}_2(U,H) \cong U^* \otimes H$ as a Hilbert space tensor product [KR97, Prop. 2.6.9]. ◊

These spaces are nested, in particular for all $q \geq 1$ we have

$$\begin{aligned} &\mathfrak{C}^p_T(V) \subset \mathfrak{C}^{p'}_T(V) \subset \mathfrak{C}^0_T(V) \quad \text{for } p > p' > 0, \\ &\mathfrak{L}^{p,q}_T(V) \subset \mathfrak{L}^{p',q}_T(V) \subset \mathfrak{L}^{0,q}_T(V) \quad \text{for } p > p' > 0 \end{aligned}$$

and

$$\mathfrak{C}_T^p(V) \subset \mathfrak{L}_T^{p,q}(V) \quad \text{for } p \geq 0\,.$$

Next we define the space of *elementary processes* as

$$\begin{aligned}\mathcal{E} = \operatorname{span}(\{\Phi \in L([0,T] \times \Omega, L(U,H)) \mid \\ \Phi(t,\omega) = \mathbb{1}_A(t,\omega) \cdot e_i \otimes f_j, A \in \mathcal{R}_T, i \in \mathbb{N}, j \in \mathbb{N}\})\end{aligned}$$

where we identify $e \otimes f \in U \otimes H$ with the rank-one operator $[U \ni u \mapsto \langle e,u\rangle_U f \in H] \in L(U,H)$. Note that from the definition of the predictable rectangles the elementary processes are adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. It is easy to see that the $\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$-norm (1.2) also defines a seminorm on $\mathcal{E}$ for every $p > 0$. Taking again the quotient $\mathcal{E}^* = \mathcal{E}/_{\|\cdot\|_{\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))}}$, we see that $(\mathcal{E}^*, \|\cdot\|_{\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))})$ is a normed space. The following proposition describes the completion of this space.

Proposition 1.4.
The space of elementary processes $\mathcal{E}^*$ is dense in $\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$ for every $p > 0$. In particular, we have

$$\overline{\mathcal{E}^*} = \mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$$

where the completion is with respect to the $\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$-norm.

Proof. From $\mathcal{E}^* \subset \mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$ and the completeness of $\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$ it follows that $\overline{\mathcal{E}^*} \subseteq \mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$.

To prove the converse inclusion let $\Phi \in \mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$. From [Coh13, Prop. E.2] we get the existence of a sequence $(\Phi_n)_{n\in\mathbb{N}} \colon [0,T]\times\Omega \to \mathrm{L}_2(U,H)$ of simple, $\mathrm{L}_2(U,H)$-valued functions converging pointwise to Φ. Since the set $\{e_i \otimes f_j \mid i \in \mathbb{N}, j \in \mathbb{N}\} \subset U \otimes H$ is an orthonormal basis of $U \otimes H$ [KR97, Thm. 2.6.4] its span is dense in $U \otimes H \cong \mathrm{L}_2(U,H)$. Thus, we can also find a sequence $(\tilde{\Phi}_n)_{n\in\mathbb{N}} \subset \mathcal{E}^*$ of elementary processes converging pointwise to Φ. Two applications of Lebesgue's dominated convergence theorem show that this sequence converges also in the $\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))$-norm to Φ. Thus $\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H)) \subseteq \overline{\mathcal{E}^*}$. ☐

Now we are able to define a stochastic integral $I \colon \mathcal{E} \to \mathfrak{C}_T^2(H)$ for $\Phi(t,\omega) = \mathbb{1}_A(t,\omega) \cdot e_i \otimes f_j$ by

$$I\Phi(t) = \begin{cases} \mathbb{1}_{F_{s_1}}(\omega) \cdot [\beta_i(s_2 \wedge t) - \beta_i(s_1 \wedge t)] \cdot f_j & \text{if } A = (s_1, s_2] \times F_{s_1} \in \mathcal{R}_T, \\ 0 & \text{if } A = \{0\} \times F_0 \in \mathcal{R}_T \end{cases}$$

and extending linearly to all of $\mathcal{E}$. We also write $\int_0^t \Phi_s \,\mathrm{d}W_s = I\Phi(t)$. The integral is also well-defined on $\mathcal{E}^*$ as for every $\Phi \in \mathcal{E}$ it holds that $\|\Phi\|_{\mathfrak{L}_T^{2,2}(\mathrm{L}_2(U,H))} = 0$ implies $\|I\Phi\|_{\mathfrak{C}_T^2(H)} = 0$. Thus, we can define the integral of an equivalence class in $\mathcal{E}^*$ as the integral of an arbitrary element of that equivalence class.

Examining now the properties of this integral, we first observe that the integral process is continuous. This follows from the continuity of the real-valued Wiener processes.

Furthermore, it is a martingale, since for $\Phi(t,\omega) = \mathbb{1}_{(s_1,s_2]\times A}(t,\omega)\cdot e_i \otimes f_j$ and $s_1 < s < t$ we have $A \in \mathcal{F}_s$ and

$$\begin{aligned}\mathbb{E}[I\Phi(t) \mid \mathcal{F}_s] &= \mathbb{E}[\mathbb{1}_A(\omega)\cdot[\beta_i(s_2 \wedge t) - \beta_i(s_1 \wedge t)]\cdot f_j \mid \mathcal{F}_s] \\ &= \mathbb{1}_A(\omega)\cdot[\beta_i(s_2 \wedge s) - \beta_i(s_1 \wedge s)]\cdot f_j \\ &= I\Phi(s)\end{aligned}$$

and $\mathbb{E}[I\Phi(t) \,|\, \mathcal{F}_s] = 0 = I\Phi(s)$ for $s \le s_1$ because of the martingale property of the scalar Wiener process. The martingale property also holds trivially if $\Phi(t,\omega) = \mathbb{1}_{\{0\}\times A}(t,\omega)\cdot e_i \otimes f_j$ with $A \in \mathcal{F}_0$. This also shows that $\mathbb{E}[I\Phi(t)] = 0$ for all $t \in [0,T]$. By construction the integral is a linear operator, and it is even a bounded operator as can be seen by considering a general element

$$\Phi_t(\omega) = \sum_{k=1}^{K}\sum_{l=1}^{L_k}\sum_{m=1}^{M_{k,l}} \mathbb{1}_{S_k\times A_k}(t,\omega)\cdot e_{i_{k,l}} \otimes f_{j_{k,l,m}}\cdot c_{k,l,m} \in \mathcal{E}$$

where $S_k \times A_k \in \mathcal{R}_T$ are pairwise disjoint, $i_{k,l} \in \mathbb{N}$ are distinct for different l and $j_{k,l,m} \in \mathbb{N}$ are distinct for different m. Indeed, we have

$$\|I\Phi\|_{\mathfrak{C}^2_T(H)} \le 2\cdot \sup_{t\in[0,T]} \|I\Phi(t)\|_{L^2(\Omega,H)}$$

by Doob's maximal inequality and

$$\begin{aligned}\sup_{t\in[0,T]} \|I\Phi(t)\|^2_{L^2(\Omega,H)} &= \|I\Phi(T)\|^2_{L^2(\Omega,H)} \\ &= \sum_{k=1}^{K} \lambda(S_k)\cdot \mathrm{P}(A_k)\cdot \sum_{l=1}^{L_k}\sum_{m=1}^{M_{k,l}} |c_{k,l,m}|^2 \qquad (1.6) \\ &= \|\Phi\|^2_{\mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,H))}\end{aligned}$$

where the first equality holds because $\left(\|I\Phi(t)\|^2_H\right)_{t\in[0,T]}$ is a submartingale, since the integral process itself is a martingale. Thus, we see that the integral is a bounded operator between $(\mathcal{E}^*, \|\cdot\|_{\mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,H))})$ and $\mathfrak{C}^2_T(H)$. Since the operator is bounded, we can extend it (as usual by considering Cauchy sequences) to the completion $\mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,H))$ of $\mathcal{E}^*$. This extension will be called the *Itô integral* of Φ.

The equality (1.6) also holds for the extended operator and we have thus proved the following important property.

Proposition 1.5 (Itô isometry).
The Itô integral seen as a linear operator

$$I\colon \mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,H)) \to L^2(\Omega,H)$$

is an isometry, i.e. for all $\Phi \in \mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,H))$ it holds

$$\|I\Phi(T)\|_{L^2(\Omega,H)} = \|\Phi\|_{\mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,H))}\,.$$

Remark 1.6. In view of Definition 1.1 of a cylindrical Wiener process we can alternatively define the Itô integral directly for $\Phi \in \mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,H))$ by

$$I\Phi(t) = \int_0^t \Phi_s \,\mathrm{d}W_s \coloneqq \sum_{n\in\mathbb{N}} \int_0^t \Phi_s e_n \,\mathrm{d}\beta_n(s)$$

where the integrals are then standard Itô integrals against real-valued Wiener processes. The sum converges in $L^2(\Omega, H)$ due to the assumption on Φ. ◊

1.1.2 L^0 Theory

Through a standard localisation argument we can extend the stochastic integral to all processes in $\mathfrak{L}^{0,2}_T(\mathrm{L}_2(U,H))$ [NVW07, Section 5]. We then obtain a bounded linear operator [NVW07, Theorems 5.5, 5.9]

$$I\colon \mathfrak{L}^{0,2}_T(\mathrm{L}_2(U,H)) \to \mathfrak{C}^0_T(H)\,.$$

Definition 1.7 (Itô process)**.**
Let $\xi \in \mathbb{M}(\mathcal{F}_0, \mathcal{B}(H))$, $\varphi \in \mathfrak{L}^{0,1}_T(H)$ *and* $\Phi \in \mathfrak{L}^{0,2}_T(\mathrm{L}_2(U,H))$*. Then the process* $X \in \mathfrak{C}^0_T(H)$ *defined by*

$$X_t = \xi + \int_0^t \varphi_s \,\mathrm{d}s + \int_0^t \Phi_s \,\mathrm{d}W_s, \quad t \in [0,T]$$

is well-defined and called an H*-valued* Itô process*.*

For the class of Itô processes we have the following fundamental Itô formula. It is the analogue of the deterministic chain rule for stochastic processes and is a cornerstone of stochastic analysis.

Theorem 1.8 (Itô formula)**.**
Let $X \in \mathfrak{C}^0_T(H)$ *be an Itô process with a representation as in Definition* 1.7*,* V *another separable Hilbert space and* $f \in C^{1,2}([0,T]\times H, V)$*. Then the stochastic process defined by* $Y_t = f(t, X_t)$ *is a* V*-valued Itô process belonging to* $\mathfrak{C}^0_T(V)$*, and we have* P*-a.s. for all* $t \in [0,T]$

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t (\partial_1 f)(s,X_s)\,\mathrm{d}s + \int_0^t (\partial_2 f)(s,X_s)\varphi_s \,\mathrm{d}s \\ &\quad + \int_0^t (\partial_2 f)(s,X_s)\Phi_s \,\mathrm{d}W_s + \frac{1}{2}\int_0^t \sum_{n\in\mathbb{N}} (\partial_2^2 f)(s,X_s)(\Phi_s e_n, \Phi_s e_n)\,\mathrm{d}s \end{aligned}$$

and all integrals are well-defined.

A proof can be found in [Brz+08, Theorem 2.4].

1.1.3 L^p Theory

Suppose we know for $p \neq 2$ that our integrand is in $\mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,H)) \subseteq \mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,H))$. Do we then also get estimates for the L^p norms of the Itô integral? As the next proposition shows, the Itô integral defines a bounded linear operator on the subspaces $\mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,H)) \subset \mathfrak{L}^{0,2}_T(\mathrm{L}_2(U,H))$ for all $p > 0$.

Proposition 1.9 (Burkholder–Davis–Gundy inequality). *Let $p > 0$. The Itô integral seen as a linear operator*

$$I\colon \mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,H)) \to \mathfrak{C}^p_T(H)$$

is bounded. In particular, there exists $\tau_p > 0$ such that for all $\Phi \in \mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,H))$ it holds

$$\|I\Phi\|_{\mathfrak{C}^p_T(H)} \leq \tau_p \cdot \|\Phi\|_{\mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,H))} .$$

Proof. For the proof see [DZ14, Theorem 4.36]. The case $p \geq 2$ can be proved using the Itô formula (Theorem 1.8), Hölder's inequality and Doob's maximal inequality. The case $p \in (0,2)$ can be proved using Lenglart's inequality. □

We will often need the following simple corollary to bound the L^p norm of stochastic integrals. Note that the second estimate follows from Minkowski's integral inequality (see e.g. [Hyt+16, Proposition 1.2.22]).

Corollary 1.10. *Let $p > 0$ and $t \in [0,T]$. For all $\Phi \in \mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,H))$ it holds*

$$\|I\Phi(t)\|_{L^p(\Omega,H)} \leq \tau_p \cdot \|\Phi\|_{\mathfrak{L}^{p,2}_t(\mathrm{L}_2(U,H))} .$$

If $p \geq 2$ we additionally have

$$\|I\Phi(t)\|_{L^p(\Omega,H)} \leq \tau_p \cdot \left(\int_0^t \|\Phi_s\|^2_{L^p(\Omega,\mathrm{L}_2(U,H))} \,\mathrm{d}s\right)^{\frac{1}{2}} .$$

Let us now extend a simple property of the stochastic integral of elementary processes to a wider class of integrands.

Proposition 1.11. *Let $p \geq 1$ and $t \in [0,T]$. For all $\Phi \in \mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,H))$ it holds*

$$\mathbb{E}[I\Phi(t)] = 0.$$

Proof. Since $p \geq 1$ and by Corollary 1.10 we have

$$\|I\Phi(t)\|_{L^1(\Omega,H)} \leq \|I\Phi(t)\|_{L^p(\Omega,H)} \leq \tau_p \|\Phi\|_{\mathfrak{L}^{p,2}_t(\mathrm{L}_2(U,H))} < \infty \tag{1.7}$$

and so $I\Phi(t)$ is integrable. Let now $(\Phi_n)_{n\in\mathbb{N}} \subset \mathcal{E}^*$ be a Cauchy sequence of elementary processes converging to Φ in $\mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,H))$. Then Corollary 1.10 shows that also $(I\Phi_n(t))_{n\in\mathbb{N}}$ converges to $I\Phi(t)$ in $L^p(\Omega,H)$. Thus, there exists an almost surely convergent subsequence $(I\Phi_{n_k}(t))_{k\in\mathbb{N}} \subset (I\Phi_n(t))_{n\in\mathbb{N}}$. Since $I\Phi(t)$ is integrable we can apply the dominated convergence theorem [Coh13, Thm. E.6] to get that

$$\mathbb{E}[I\Phi(t)] = \mathbb{E}\left[\lim_{k\to\infty} I\Phi_{n_k}(t)\right] = \lim_{k\to\infty} \mathbb{E}[I\Phi_{n_k}(t)] = 0.$$

The last equality holds as we have seen before that the stochastic integral of elementary processes has zero mean. □

Requiring the p-integrability of the initial value and the integrands, we can define a refined notion of Itô process.

Definition 1.12 (L^p Itô process)**.**
Let $p \geq 0$, $\xi \in L^p(\Omega, H)$ $\mathcal{F}_0$*-measurable,* $\varphi \in \mathfrak{L}_T^{p,1}(H)$ *and* $\Phi \in \mathfrak{L}_T^{p,2}(\mathrm{L}_2(U, H))$*. Then the process* $X \in \mathfrak{C}_T^p(H)$ *defined by*

$$X_t = \xi + \int_0^t \varphi_s \, \mathrm{d}s + \int_0^t \Phi_s \, \mathrm{d}W_s, \quad t \in [0, T]$$

is well-defined and called an H*-valued* p-integrable (or L^p) Itô process*.*

That for $p > 0$ the process X is indeed in $\mathfrak{C}_T^p(H)$ is easily seen using the (modified) triangle inequality (1.5) and Proposition 1.9. In particular, we have

$$\|X\|_{\mathfrak{C}_T^p(H)} \leq \kappa_p^2 \left(\|\xi\|_{L^p(\Omega,H)} + \|\varphi\|_{\mathfrak{L}_T^{p,1}(H)} + \tau_p \|\Phi\|_{\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H))} \right) . \tag{1.8}$$

The following lemma shows that the class of p-integrable Itô processes is stable under the application of functions satisfying a polynomial growth assumption.

Lemma 1.13.
Let $p \geq 0$, $r \geq 0$, $X \in \mathfrak{C}_T^{rp}(H)$ *an* L^{rp} *Itô process,* V *another separable Hilbert space and* $f \in C^{1,2}([0, T] \times H, V)$*. Suppose that* f *has polynomial growth of order* r *in the sense that there exists* $K > 0$ *such that for all* $(t, x) \in [0, T] \times H$ *we have*

$$\|f(t, x)\|_V \leq K \left(1 + \|x\|_H^r\right) .$$

Then the stochastic process given by $Y_t = f(t, X_t)$ *is a* V*-valued* p*-integrable Itô process and for* $p > 0$ *we have*

$$\|Y\|_{\mathfrak{C}_T^p(V)} \leq K \cdot \kappa_p (1 + \|X\|_{\mathfrak{C}_T^{rp}(H)}^r) \, . \tag{1.9}$$

Proof. That Y is indeed an Itô process follows from an application of the Itô formula (Theorem 1.8). The estimate (1.9) follows from the polynomial growth assumption and the (modified) triangle inequality (1.5). ☐

Example 1.14. The polynomial growth assumption is for example fulfilled for Lipschitz functions. In this case we can choose $r = 1$. $\Diamond$

1.2 Stochastic Evolution Equations

In this work we are concerned with semi-linear parabolic stochastic evolution equations (SEEs) of the form

$$\begin{cases} \mathrm{d}X_t = [AX_t + F(X_t)] \, \mathrm{d}t + B(X_t) \, \mathrm{d}W_t, & t \in (0, T], \\ X_0 = \xi \end{cases} \tag{SEE}$$

on a separable Hilbert space H. Let U be another separable Hilbert space with orthonormal basis $(e_n)_{n \in \mathbb{N}} \subset U$. Here, $A \colon \mathrm{D}(A) \subseteq H \to H$ is a linear operator, $F \colon H \to H$ and

$B\colon H \to \mathrm{L}_2(U,H)$ are possibly non-linear functions and $T > 0$ is the final time. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Then $(W_t)_{t\in[0,T]}$ is assumed to be a cylindrical Wiener process on U (see Definition 1.1) and $\xi \in \mathbb{M}(\mathcal{F}_0, \mathcal{B}(H))$ is the initial value.

1.2.1 The Semigroup Approach

For $\delta \in (0, \pi]$ denote by

$$\Sigma_\delta := \{\lambda \in \mathbb{C} \mid |\arg \lambda| < \delta\} \setminus \{0\}$$

the complex open sector opening to the right.

Definition 1.15 (Sectorial operator, see [EN00, Definition 4.1])**.**
A closed operator $(A, \mathrm{D}(A))$ on a Hilbert space H is called sectorial *with angle δ if there exists $\delta \in (0, \pi]$ such that*

i) $\Sigma_\delta \subset \rho(A)$

and if for each $\delta' \in (0, \delta)$ exists a constant $M_{\delta'} \geq 1$ such that

ii) $\|R(\lambda, A)\| \leq \frac{M_{\delta'}}{|\lambda|}$ *for all* $\lambda \in \overline{\Sigma}_{\delta'} \setminus \{0\}$.

Definition 1.16 (Analytic semigroup, see [EN00, Definition 4.5])**.**
Let $\delta \in (0, \frac{\pi}{2}]$. A family of operators $T = (T(z))_{z\in\Sigma_\delta\cup\{0\}} \subset L(H,H)$ is called an analytic semigroup *of angle δ if*

i) $T(0) = \mathrm{Id}_H$ *and* $T(z_1)T(z_2) = T(z_1 + z_2)$ *for all* $z_1, z_2 \in \Sigma_\delta$,

ii) the mapping $T\colon \Sigma_\delta \to L(H,H)$, $z \mapsto T(z)$ *is analytic and*

iii) $\lim\limits_{\Sigma_{\delta'} \ni z \to 0} T(z)x = x$ *for all* $x \in H$ *and* $\delta' \in (0, \delta)$.

The analytic semigroup T is called bounded *if we additionally have that*

iv) $z \mapsto \|T(z)\|$ *is bounded in* $\Sigma_{\delta'}$ *for every* $\delta' \in (0, \delta)$.

Theorem 1.17 (see [EN00, Theorem 4.6])**.**
Let $\delta \in (0, \frac{\pi}{2}]$. For a closed operator $(A, \mathrm{D}(A))$ on a Hilbert space H the following statements are equivalent.

a) A is densely defined and sectorial with angle $\frac{\pi}{2} + \delta$.

b) A generates a bounded analytic semigroup T on H of angle δ.

Let now A be a densely defined sectorial operator with angle $\pi/2 + \delta$ for $\delta \in (0, \pi/2]$ and let $(e^{tA})_{t\in[0,T]} \subset L(H,H)$ be the analytic semigroup generated by A. Then we can define so-called mild solutions.

Definition 1.18 (Mild solution).
A process $X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$ *is called a* mild solution *of the equation* (SEE) *if*

i) $\forall t \in (0,T]$: $\left[s \mapsto e^{(t-s)A}F(X_s)\right] \in \mathfrak{L}_t^{0,1}(H)$,

ii) $\forall t \in (0,T]$: $\left[s \mapsto e^{(t-s)A}B(X_s)\right] \in \mathfrak{L}_t^{0,2}(\mathrm{L}_2(U,H))$,

iii) $X_0 = \xi$ P*-a.s. and*

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_s$$

holds P*-a.s. for all* $t \in (0,T]$.

It turns out that for a sectorial operator A mild solutions exist even in cases where the functions F and B are not H-valued, respectively $\mathrm{L}_2(U,H)$-valued, but take values in a larger Hilbert space that depends on the operator A. This is a consequence of the excellent smoothing properties of the analytic semigroup generated by A. In particular, these larger Hilbert spaces can be characterised as the domains of fractional powers of $-A$.

Definition 1.19 (Fractional powers).
Let $\alpha > 0$. *Assume furthermore that* $0 \in \rho(A)$. *This ensures that there exists* $\lambda < 0$ *such that* $\sup_{\mu \in \sigma(A)} \mathrm{Re}(\mu) \leq \lambda$. *We then define the negative fractional powers as the bounded linear operators* $(-A)^{-\alpha} \in L(H,H)$ *satisfying*

$$(-A)^{-\alpha} := \frac{1}{2\pi i}\int_\gamma z^{-\alpha}R(z,-A)\,\mathrm{d}z.$$

Here we define $z^{-\alpha} = e^{-\alpha \log z}$ *with the principal branch of the logarithm which is holomorphic on the slit plane* $\mathbb{C}\setminus(-\infty,0]$ *and* γ *is a positively oriented (i.e. counter-clockwise) path around the spectrum of* $-A$ *avoiding the branch cut at the negative real axis. Furthermore, we define the positive fractional powers as the densely-defined, closed linear operators* $((-A)^\alpha, \mathrm{D}((-A)^\alpha))$ *satisfying*

$$(-A)^\alpha := \left((-A)^{-\alpha}\right)^{-1}$$

with $\mathrm{D}((-A)^\alpha) = (-A)^{-\alpha}(H) \subset H$. *Finally, we define* $(-A)^0 = \mathrm{Id}_H$.

For basic properties concerning the fractional powers we refer to [EN00, Section II.5.c] and [Paz83, Section 2.6].

Definition 1.20 (Interpolation spaces associated to A).
As the interpolation spaces associated to (H,A) *we define the family of Hilbert spaces* $(H_\alpha, \|\cdot\|_{H_\alpha})_{\alpha\in\mathbb{R}}$ *by* $H_0 = H$, $H_\alpha = \mathrm{D}((-A)^\alpha)$ *for* $\alpha > 0$ *and* $H_\alpha = \overline{H}^{\|\cdot\|_{H_\alpha}}$ *for* $\alpha < 0$ *where* $\|\cdot\|_{H_\alpha} := \|(-A)^\alpha\cdot\|_H$ *for all* $\alpha \in \mathbb{R}$.

Remark 1.21. Note that, a priori, the norm on H_α for $\alpha < 0$ is given by the continuous extension of the norm $\|\cdot\|_{H_\alpha}$, which is first only defined for elements of H. Only after we know that A can be continuously extended, see Proposition 1.23, can we identify the norm as $\|h\|_{H_\alpha} := \|(-A)^\alpha h\|_H$ for all $h \in H_\alpha$. ◊

Remark 1.22. The term interpolation space is justified, because these spaces are intermediate spaces between $\mathrm{D}(A)$ and H in the terminology of interpolation theory. Indeed, in some special cases (e.g. if A admits a bounded H^∞ functional calculus [Wei06] or more generally if $-A + \mathrm{Id}_H$ has bounded imaginary powers) we even have $H_\alpha = [H, \mathrm{D}(A)]_\alpha$ with equivalent norms for $0 < \alpha < 1$ (see [Yag10, Theorem 16.3] or [Haa06, Theorem 6.6.9]). ♢

The following properties and more can be found in the detailed Chapter V of [Ama95]. In particular, we refer to [Ama95, Theorem V.1.4.12]. A more modern account of this theory, also focusing on the differences between homogeneous and inhomogeneous spaces and the implication of the assumption $0 \in \rho(A)$, can be found in [KW04, Section 15].

Proposition 1.23.
Let $(H_\alpha, \|\cdot\|_{H_\alpha})_{\alpha\in\mathbb{R}}$ be the family of interpolation spaces associated to (H, A). Fix $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha > \beta$. Then

a) $H_\alpha \hookrightarrow H_\beta$ densely and continuously, i.e. $\|\mathrm{Id}_{H_\alpha}\|_{L(H_\alpha, H_\beta)} < \infty$,

b) $H_{\gamma+\alpha} \cong H_\gamma$ with the isometric isomorphism given by $A^\alpha \colon H_{\gamma+\alpha} \to H_\gamma$ and

c) $H_\alpha \cong V^*_{-\alpha}$, where $(V_\alpha, \|\cdot\|_{V_\alpha})_{\alpha\in\mathbb{R}}$ is the family of interpolation spaces associated to the adjoint operator (H^*, A^*) and the duality pairing is given by the canonical pairing between H and H^*.

Furthermore, if for $\theta \in (0, 1)$ we define $\gamma = \theta\alpha + (1-\theta)\beta \in (\beta, \alpha)$. Then there exists $C = C(\alpha, \beta, \gamma) \in [1, \infty)$ such that

d) $\|x\|_{H_\gamma} \le C\|x\|^\theta_{H_\alpha}\|x\|^{1-\theta}_{H_\beta}$ for all $x \in H_\alpha$.

Now we can come back to the smoothing properties of the semigroup alluded to before. The scale of interpolation spaces that we just defined allows us to measure smoothness in terms of our dominating operator. We will see that already after an arbitrarily short amount of time the semigroup maps everything into H_α for each $\alpha > 0$, thus immediately providing arbitrary smoothness. A popular instance of this effect is the fact that every solution to the heat equation is infinitely differentiable after a positive time. Here the heat semigroup is given by the convolution with a Gaussian and is generated by the Laplace operator.

A proof of the following theorem can be found, e.g., in [Paz83, Theorem 6.13] and [RR04, Lemma 12.36].

Theorem 1.24.
Assume again that $0 \in \rho(A)$. Then there exist constants $C_\alpha \in [1, \infty)$, $\alpha \in (0, \infty)$ such that for all positive times $t > 0$ the following estimates hold.

a) For $\alpha \in (0, \infty)$ we have $e^{tA}(H) \subset H_\alpha$ and it holds $\left\|e^{tA}\right\|_{L(H, H_\alpha)} \le C_\alpha \cdot t^{-\alpha}$.

b) For $\alpha \in (0, 1]$ it holds $\left\|e^{tA} - \mathrm{Id}_{H_\alpha}\right\|_{L(H_\alpha, H)} \le C_\alpha \cdot t^\alpha$.

Remark 1.25. Since the fractional powers commute with the semigroup and using Proposition 1.23 we can see that actually $e^{tA}(H_\alpha) \subset H_\beta$ for all $\alpha, \beta \in \mathbb{R}$ and $t > 0$. We also have estimates $\|e^{tA}\|_{L(H_\gamma, H_{\gamma+\alpha})} \leq C_\alpha \cdot t^{-\alpha}$ and $\|e^{tA} - \mathrm{Id}_{H_{\gamma+\alpha}}\|_{L(H_{\gamma+\alpha}, H_\gamma)} \leq C_\alpha \cdot t^\alpha$ for all $\gamma \in \mathbb{R}$ and $\alpha \in (0, \infty)$ and $\alpha \in (0, 1]$, respectively. ♢

Remark 1.26. In the case that $0 \notin \rho(A)$ we can rescale the semigroup and obtain analogous estimates. In particular, let $\lambda > 0$. Then the operator $(A - \lambda \mathrm{Id}_H, \mathrm{D}(A))$ also generates a bounded analytic semigroup given by $e^{t(A-\lambda \mathrm{Id}_H)} = e^{-\lambda t} e^{tA}$ and has $0 \in \rho(A - \lambda \mathrm{Id}_H)$. See [EN00, Pragraph II.2.2] for further information regarding this rescaling procedure. Thus, we can define the fractional powers and the interpolation spaces in terms of $A - \lambda \mathrm{Id}_H$. This leads to estimates of the semigroup associated to A with an additional factor of $e^{\lambda t}$. In particular, we then have $\|e^{tA}\|_{L(H, H_\alpha)} \leq C_\alpha \cdot t^{-\alpha} \cdot e^{\lambda t}$ and $\|e^{tA} - \mathrm{Id}_{H_\alpha}\|_{L(H_\alpha, H)} \leq C_\alpha \cdot t^\alpha \cdot e^{\lambda t}$ for all $\alpha \in (0, \infty)$ and $\alpha \in (0, 1]$, respectively. On a finite time horizon $[0, T]$ these can be estimated by a constant, $e^{\lambda T}$. ♢

Remark 1.27. We refer to [Lun95, Chapter 2] for a more general class of interpolation spaces in which similar estimates to Theorem 1.24 can be seen to hold. ♢

The next theorem provides sufficient conditions for the existence and uniqueness of mild solutions. It is a special case of [DJR19, Proposition 2].

Theorem 1.28 (Existence and uniqueness of mild solutions)**.**
Consider the equation (SEE)*. Let* $p \geq 2$*,* $\alpha > -1$ *and* $\beta > -\frac{1}{2}$*. Assume that*

i) A *is a densely defined sectorial operator,*

ii) $\xi \in L^p(\Omega, H)$*,*

iii) $F \in \mathrm{Lip}(H, H_\alpha)$ *and*

iv) $B \in \mathrm{Lip}(H, \mathrm{L}_2(U, H_\beta))$*.*

Then there exists an up to modifications unique mild solution X *of* (SEE)*. Furthermore, it holds that* $\sup_{t \in [0,T]} \|X_t\|_{L^p(\Omega, H)} < \infty$*.*

1.2.2 Mild Stochastic Calculus

In [DJR19] Da Prato, Jentzen and Röckner developed a new framework of so-called mild stochastic calculus tailored to work with mild solutions of SPDEs. Its main tools are a notion of mild Itô process and a corresponding mild Itô formula. A generalisation to certain UMD Banach spaces was given by Cox et al. in [Cox+18].

Consider two additional separable Hilbert spaces $\check{H} \subseteq H \subseteq \hat{H}$ with continuous and dense embeddings and let $\Lambda = \{(s, t) \in [0, T]^2 \mid s < t\}$.

In the following we need the concept of an *evolution family* or two-parameter semigroup (see also [EN00, Definition 9.2]). Let $\mathcal{S}(\hat{H}, \check{H})$ denote the Borel σ-algebra generated by the strong operator topology on $L(\hat{H}, \check{H})$.

Definition 1.29 (Evolution family)**.**
A mapping $S \in \mathbb{M}(\mathcal{B}(\Lambda), \mathcal{S}(\hat{H}, \check{H}))$ *is called a* (strongly continuous) evolution family *if*

i) $S_{r,t}S_{s,r} = S_{s,t}$ *for all* $0 \leq s < r < t \leq T$ *and*

ii) $S\colon \Lambda \to L(\hat{H}, \check{H})$ *is continuous w.r.t. the strong operator topology on* $L(\hat{H}, \check{H})$ *(strongly continuous).*

Definition 1.30 (Mild L^p Itô process)**.**
Let $p \geq 0$, $\xi \in \mathbb{M}(\mathcal{F}_0, \mathcal{B}(H))$, $\varphi \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(\hat{H}))$, $\Phi \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(\mathrm{L}_2(U, \hat{H})))$ *and* $S \in \mathbb{M}(\mathcal{B}(\Lambda), \mathcal{S}(\hat{H}, \check{H}))$ *an evolution family. Suppose that*

i) $\xi \in L^p(\Omega, H)$,

ii) $\forall t \in (0,T]\colon$ $[s \mapsto S_{s,t}\varphi_s] \in \mathfrak{L}^{p,1}_t(\check{H})$ *and*

iii) $\forall t \in (0,T]\colon$ $[s \mapsto S_{s,t}\Phi_s] \in \mathfrak{L}^{p,2}_t(\mathrm{L}_2(U, \check{H}))$.

Then the process $X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$ *defined by* $X_0 = \xi$ *and*

$$X_t = S_{0,t}\xi + \int_0^t S_{s,t}\varphi_s \,\mathrm{d}s + \int_0^t S_{s,t}\Phi_s \,\mathrm{d}W_s, \quad t \in (0,T]$$

is well-defined and called a mild (L^p) Itô process*. Furthermore, we have for all* $t \in [0,T]$ *that* $X_t \in L^p(\Omega, H)$.

These mild Itô processes are a generalisation of the standard Itô processes defined in Definition 1.7 and Definition 1.12. To see this, choose $\hat{H} = \check{H} = H$ and $S \equiv \mathrm{Id}_H$. Then every Itô process is also a mild Itô process. We also see now that, thanks to the regularisation provided by the analytic semigroup generated by A, every mild solution of (SEE) (see Definition 1.18) is a mild Itô process with $\hat{H} = H_{\min\{\alpha,\beta,0\}}$, $\check{H} = H_r$ and $S_{s,t} = e^{(t-s)A} \in L(\hat{H}, \check{H})$ for every $(s,t) \in \Lambda$ and $r \in [0, 1+\alpha, \frac{1}{2}+\beta)$.

The result in [Cox+18, Lemma 3.3] shows that a mild Itô process is at each positive time almost surely $\check{H}$-valued. Thus, in the following we will usually work with this (for $t \in (0,T]$) $\check{H}$-valued modification. In particular, if additionally $\xi \in L^p(\Omega, \check{H})$ then

$$\sup_{t \in [0,T]} \|X_t\|_{L^p(\Omega, \check{H})} < \infty\,.$$

If we make the simpler assumptions that for $p > 0$ we have $\xi \in L^p(\Omega, \check{H})$, $\varphi \in \mathfrak{L}^{p,1}_T(\check{H})$, $\Phi \in \mathfrak{L}^{p,2}_T(\mathrm{L}_2(U, \check{H}))$ and $C_0 \coloneqq \sup_{(s,t)\in\Lambda} \|S_{s,t}\|_{L(H,H)} < \infty$ then the conditions of Definition 1.30 are naturally fulfilled and we have

$$\sup_{t \in [0,T]} \|X_t\|_{L^p(\Omega,\check{H})} \leq C_0 \cdot \kappa_p^2 \left(\|\xi\|_{L^p(\Omega,\check{H})} + \|\varphi\|_{\mathfrak{L}^{p,1}_T(\check{H})} + \tau_p \cdot \|\Phi\|_{\mathfrak{L}^{p,2}_T(\mathrm{L}_2(U,\check{H}))} \right) < \infty\,. \tag{1.10}$$

We can now formulate the following mild Itô formula (see [DJR19, Theorem 1] and [Cox+18, Theorem 3.5]).

Theorem 1.31 (Mild Itô formula)**.**
Let $X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$ *be a mild Itô process with a representation as in Definition* 1.30*,*

V another separable Hilbert space and $f \in C^{1,2}([0,T] \times \check{H}, V)$. Then we have P*-a.s. for all $t \in (0,T]$*

$$
\begin{aligned}
f(t, X_t) = f(0, S_{0,t}X_0) &+ \int_0^t (\partial_1 f)(s, S_{s,t}X_s)\,\mathrm{d}s \\
&+ \int_0^t (\partial_2 f)(s, S_{s,t}X_s)S_{s,t}\varphi_s\,\mathrm{d}s + \int_0^t (\partial_2 f)(s, S_{s,t}X_s)S_{s,t}\Phi_s\,\mathrm{d}W_s \\
&+ \frac{1}{2}\int_0^t \sum_{n\in\mathbb{N}} (\partial_2^2 f)(s, S_{s,t}X_s)(S_{s,t}\Phi_s e_n, S_{s,t}\Phi_s e_n)\,\mathrm{d}s
\end{aligned}
$$

and all integrals are well-defined.

We will often need to estimate the norm of functions of stochastic processes. In the case of functions with polynomial growth we can ensure under mild assumptions that the stochastic integral in the Itô formula is a martingale and thus has vanishing expectation. This result is provided by the following consequence of the mild Itô formula.

Proposition 1.32.
Let $p \geq 0$, $X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$ be a mild L^p Itô process with a representation as in Definition 1.30*, V another separable Hilbert space and $f \in C^2(\check{H}, V)$. Suppose that f has polynomial growth of order p in the sense that there exists $K > 0$ such that for all $x \in \check{H}$ we have*

$$\|f(x)\|_V \leq K\left(1 + \|x\|_{\check{H}}^p\right).$$

Then the stochastic process given by $Y_t = f(X_t)$ is a V-valued integrable Itô process and we have

$$
\begin{aligned}
\|\mathbb{E}[f(X_T)]\|_V \leq \|\mathbb{E}[f(S_{0,T}X_0)]\|_V &+ \int_0^T \mathbb{E}\Big[\Big\| f'(S_{s,T}X_s)S_{s,T}\varphi_s \\
&+ \frac{1}{2}\sum_{n\in\mathbb{N}} f''(S_{s,T}X_s)(S_{s,T}\Phi_s e_n, S_{s,T}\Phi_s e_n)\Big\|_V\Big]\,\mathrm{d}s\,.
\end{aligned}
$$

A proof can be found in [Cox+18, Prop. 4].

1.2.3 Strong Solutions and the Kolmogorov Backward Equation

Definition 1.33 (Strong solution)**.**
A process $X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$ is called a strong solution *of the equation* (SEE) *if*

i) $X_t \in \mathrm{D}(A)$ $\lambda|_{[0,T]} \otimes$ P*-a.s.,*

ii) $[s \mapsto AX_s + F(X_s)] \in \mathfrak{L}_T^{0,1}(H)$,

iii) $[s \mapsto B(X_s)] \in \mathfrak{L}_T^{0,2}(\mathrm{L}_2(U,H))$,

iv) $X_0 = \xi$ P*-a.s. and*

$$X_t = \xi + \int_0^t AX_s + F(X_s)\,\mathrm{d}s + \int_0^t B(X_s)\,\mathrm{d}W_s \tag{1.11}$$

holds P*-a.s. for all $t \in (0,T]$.*

Thus a strong solution to (SEE) defines an Itô process. In particular, the Itô formula (Theorem 1.8) is applicable to the solution process X.

The next lemma gives a condition under which a mild Itô process is also a standard Itô process. It is proved in [CJK19, Lemma 3.2]. Remember that the linear operator A is always assumed to be densely defined and sectorial.

Lemma 1.34.
Let $p \geq 1$, $\xi \in L^p(\Omega, H_1)$, $\varphi \in \mathfrak{L}_T^{p,1}(H_1)$, $\Phi \in \mathfrak{L}_T^{p,2}(\mathrm{L}_2(U, H_1))$ *and* $S_{s,t} = e^{(t-s)A}$. *Then it holds that*

- *i)* $\sup_{(s,t)\in\Lambda} \|S_{s,t}\|_{L(H,H)} < \infty$,
- *ii) the process* $X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$ *defined by* $X_0 = \xi$ *and*

$$X_t = S_{0,t}\xi + \int_0^t S_{s,t}\varphi_s \,\mathrm{d}s + \int_0^t S_{s,t}\Phi_s \,\mathrm{d}W_s, \quad t \in (0,T]$$

  is a mild L^p *Itô process on* $\check{H} = H_1$ *and* $\hat{H} = H$ *with* $\sup_{t\in[0,T]} \|X_t\|_{L^p(\Omega,H_1)} < \infty$,
- *iii) for all* $t \in [0,T]$ *we have* $\mathrm{P}(X_t \in H_1) = 1$ *and*
- *iv) the process* X *is even a standard* L^p *Itô process in* $\mathfrak{C}_T^p(H)$ *with*

$$X_t = \xi + \int_0^t AX_s + \varphi_s \,\mathrm{d}s + \int_0^t \Phi_s \,\mathrm{d}W_s$$

  P-*a.s. for all* $t \in [0,T]$.

Remark 1.35. As we have seen in the discussion after Definition 1.30 the first three assertions are true even for $p \in (0,1)$. We suspect that the entire lemma also holds in this case. ♢

The existence of strong solutions is e.g. guaranteed by the following sufficient conditions.

Theorem 1.36 (Existence and uniqueness of strong solutions)**.**
Consider the equation (SEE). *Let* $p \geq 2$ *and assume that*

- *i) A is a densely defined sectorial operator,*
- *ii)* $\xi \in L^p(\Omega, H_1)$,
- *iii)* $F \in \mathrm{Lip}(H, H_1)$ *and*
- *iv)* $B \in \mathrm{Lip}(H, \mathrm{L}_2(U, H_1))$.

Then there exists an up to modifications unique strong solution $X \in \mathfrak{C}_T^p(H)$ *of* (SEE).

Proof. First, observe that by Theorem 1.28 the assumptions imply that there exists a mild solution X with

$$\sup_{t\in[0,T]} \|X_t\|_{L^p(\Omega,H)} < \infty\,. \tag{1.12}$$

Next, we show that the processes defined by $\varphi_s = F(X_s)$ and $\Phi_s = B(X_s)$ for $s \in [0,T]$ belong to $\mathfrak{L}_T^{p,1}(H_1)$ and $\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H_1))$, respectively. Using the Lipschitz assumption, Minkowski's integral inequality and (1.12) we obtain

$$\begin{aligned}\|\varphi\|_{\mathfrak{L}_T^{p,1}(H_1)} &= \mathbb{E}\left[\left(\int_0^T \|F(X_s)\|_{H_1}\,\mathrm{d}s\right)^p\right]^{\frac{1}{p}}\\ &\le T\cdot\|F(0)\|_{H_1} + |F|_{\mathrm{Lip}(H,H_1)}\cdot\mathbb{E}\left[\left(\int_0^T \|X_s\|_H\,\mathrm{d}s\right)^p\right]^{\frac{1}{p}}\\ &\le T\cdot\|F(0)\|_{H_1} + |F|_{\mathrm{Lip}(H,H_1)}\cdot\int_0^T \|X_s\|_{L^p(\Omega,H)}\,\mathrm{d}s\\ &\le T\cdot\|F\|_{\mathrm{Lip}(H,H_1)}\cdot\max\{1, \sup_{t\in[0,T]}\|X_t\|_{L^p(\Omega,H)}\} < \infty\end{aligned}$$

and similarly

$$\begin{aligned}\|\Phi\|_{\mathfrak{L}_T^{p,2}(\mathrm{L}_2(U,H_1))} &= \mathbb{E}\left[\left(\int_0^T \|B(X_s)\|^2_{\mathrm{L}_2(U,H_1)}\,\mathrm{d}s\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}\\ &\le T\cdot\|B(0)\|_{H_1} + |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H_1))}\cdot\mathbb{E}\left[\left(\int_0^T \|X_s\|_H\,\mathrm{d}s\right)^p\right]^{\frac{1}{p}}\\ &\le T\cdot\|B(0)\|_{H_1} + |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H_1))}\cdot\int_0^T \|X_s\|_{L^p(\Omega,H)}\,\mathrm{d}s\\ &\le T\cdot\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H_1))}\cdot\max\{1, \sup_{t\in[0,T]}\|X_t\|_{L^p(\Omega,H)}\} < \infty\,.\end{aligned}$$

Note that the application of Minkowski's inequality here is valid since we assumed $p \ge 2$.

Now the theorem follows from Lemma 1.34. In particular, the mild solution X is also a strong solution. ☐

In the next theorem we collect several results from the literature that ensure the existence of a solution to the Kolmogorov backward equation associated to (SEE) as well as provide regularity properties of this solution.

Theorem 1.37.
Assume that

- *i)* A *is a densely defined sectorial operator,*
- *ii)* $F \in \mathrm{Lip}^4(H,H_1)$ *and*
- *iii)* $B \in \mathrm{Lip}^4(H,\mathrm{L}_2(U,H_1))$.

Then there exists an up to modifications unique mild solution X^x *of* (SEE) *with initial value* $\xi = x$ *for every* $x \in H$*. Let* $\varphi \in \mathrm{Lip}^4(H,V)$ *and define the function* $u\colon [0,T]\times H \to V$ *by* $u(t,x) = \mathbb{E}[\varphi(X^x_{T-t})]$ *for all* $t \in [0,T]$, $x \in H$*. Then it holds that* *a)* $\forall x \in H_1\colon\ [t \mapsto u(t,x)] \in C^1([0,T],V)$,

b) $\forall t \in [0,T]\colon\ [x \mapsto u(t,x)] \in C_b^4(H,V)$,

c) u is a solution of the following partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = -\,\dfrac{\partial}{\partial x}u(t,x)\,(Ax+F(x)) & \\ \qquad\qquad\quad -\,\dfrac{1}{2}\displaystyle\sum_{n\in\mathbb{N}}\dfrac{\partial^2}{\partial x^2}u(t,x)\,(B(x)e_n,B(x)e_n) & t\in[0,T),\ x\in H_1\,, \\ u(T,x)=\varphi(x) & x\in H \end{cases} \tag{1.13}$$

d) and for $k \in \{1,2,3,4\}$ *and* $\delta_1,\dots,\delta_k \in [0,\frac{1}{2})$ *with* $\sum_{i=1}^k \delta_i < \frac{1}{2}$ *it holds that*

$$\begin{aligned} c_{\delta_1,\dots,\delta_k} &\coloneqq \sup_{t\in[0,T)}\sup_{x\in H}\sup_{v_1,\dots,v_k\in H\setminus\{0\}} \frac{(T-t)^{\sum_{i=1}^k\delta_i}\left\|\left(\frac{\partial^k}{\partial x^k}u\right)(t,x)(v_1,\dots,v_k)\right\|_V}{\prod_{i=1}^k\|v_i\|_{H_{-\delta_i}}} \\ &< \infty \end{aligned} \tag{1.14}$$

as well as for $\delta_1,\dots,\delta_4 \in [0,\frac{1}{2})$ *with* $\sum_{i=1}^4 \delta_i < \frac{1}{2}$

$$\begin{aligned} \tilde{c}_{\delta_1,\delta_2,\delta_3,\delta_4} &\coloneqq \sup_{t\in[0,T)}\sup_{\substack{x_1,x_2\in H\\ x_1\neq x_2}}\sup_{v_1,\dots,v_k\in H\setminus\{0\}} \frac{(T-t)^{\sum_{i=1}^4\delta_i}\left\|\left(\left(\frac{\partial^4}{\partial x^4}u\right)(t,x_1)-\left(\frac{\partial^4}{\partial x^4}u\right)(t,x_2)\right)(v_1,\dots,v_4)\right\|_V}{\|x_1-x_2\|_H\prod_{i=1}^4\|v_i\|_{H_{-\delta_i}}} \\ &< \infty. \end{aligned} \tag{1.15}$$

Equation (1.13) *is called the* Kolmogorov backward equation *associated to* (SEE).

Proof. First, the existence of a mild solution follows from Theorem 1.28. Furthermore, the assumptions ensure that for $x \in H_1$ the mild solution X^x is even a strong solution by Theorem 1.36. This is a crucial step in the proof of [CJK19, Lemma 3.3] which ensures that the function u is continuously differentiable with respect to the time variable. Differentiability with respect to the space variable is proved in [And+19, Theorem 3.3 (iii)]. That u solves (1.13) can be proved as in [DZ14, Theorem 9.25] and the bounds in d) are proved in [And+19, Theorem 3.3 (vii) and (x)]. □

Lastly, the following result is a special case of [And+19, Corollary 4.2]. It provides the basis for a mollification procedure that allows us to handle more irregular coefficient functions.

Theorem 1.38.
Let $\alpha \in [0,1)$, $\beta \in [0,\frac{1}{2})$ *and assume that*

i) A is a densely defined sectorial operator,

ii) $F \in \mathrm{Lip}^4(H, H_{-\alpha})$ and

iii) $B \in \mathrm{Lip}^4(H, \mathrm{L}_2(U, H_{-\beta}))$.

Then the smoothed coefficients $F_\varepsilon \colon H \to H_1$, $x \mapsto e^{\varepsilon A}F(x)$ and $B_\varepsilon \colon H \to \mathrm{L}_2(U, H_1)$, $x \mapsto e^{\varepsilon A}B(x)$ for $\varepsilon \in (0,T]$ satisfy the assumptions of Theorem 1.37. Denote the associated constants from (1.14) and (1.15) by $c^{(\varepsilon)}_{\delta_1,\ldots,\delta_k}$ and $\tilde{c}^{(\varepsilon)}_{\delta_1,\delta_2,\delta_3,\delta_4}$. Then for $k \in \{1,2,3,4\}$ and $\delta_1,\ldots,\delta_k \in [0,\frac{1}{2})$ with $\sum_{i=1}^k \delta_i < \frac{1}{2}$ we have

$$\sup_{\varepsilon \in (0,T]} c^{(\varepsilon)}_{\delta_1,\ldots,\delta_k} < \infty \tag{1.16}$$

and

$$\sup_{\varepsilon \in (0,T]} \tilde{c}^{(\varepsilon)}_{\delta_1,\delta_2,\delta_3,\delta_4} < \infty. \tag{1.17}$$

1.3 Numerical Approximation of Stochastic Evolution Equations

We are interested in numerical approximations of the mild solution of equation (SEE). That is, for a linear operator A and the parameters ξ, F and B all satisfying the assumptions of Theorem 1.28 we want to approximate

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_s \tag{1.18}$$

on a finite time horizon $t \in [0,T]$. As we have seen, X defines a mild Itô process (with $S_{s,t} = e^{(t-s)A}$ for $(s,t) \in \Lambda$) and thus we can use the mild Itô formula to derive heuristic approximations for functions of X. We do this by truncating the mild Itô formula and omitting the higher order terms. In particular, we employ the following two approximations for a function $f \in C^{1,2}([0,T] \times \check{H}, V)$ and $(s,t) \in \Lambda$

$$f(t, X_t) \approx f(s, e^{(t-s)A}X_s) \tag{1.19}$$

and

$$f(t, X_t) \approx f(s, e^{(t-s)A}X_s) + \int_s^t (\partial_2 f)(u, e^{(t-u)A}X_u)e^{(t-u)A}B(X_u)\,\mathrm{d}W_u\,. \tag{1.20}$$

One of the simplest schemes is the exponential Euler scheme. It is obtained by using the approximation (1.19) for both integrands in (1.18) and then employing the further approximation $e^{(t-s)A} \approx \mathrm{Id}_H$ for small time steps. We get for $(s,t) \in \Lambda$

$$\begin{aligned} X_t &\approx e^{(t-s)A}X_s + \int_s^t e^{(t-s)A}F(e^{(r-s)A}X_s)\,\mathrm{d}r + \int_s^t e^{(t-s)A}B(e^{(r-s)A}X_s)\,\mathrm{d}W_r \\ &\approx e^{(t-s)A}X_s + \int_s^t e^{(t-s)A}F(X_s)\,\mathrm{d}r + \int_s^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_r \\ &\approx e^{(t-s)A}\left[X_s + F(X_s)\cdot(t-s) + \int_s^t B(X_s)\,\mathrm{d}W_s\right]. \end{aligned}$$

Let us fix for $N \in \mathbb{N}$ an equidistant grid of $N+1$ time points of the interval $[0,T]$ denoted by $0 = t_0, \dots, t_N = T$ with $t_i = i \cdot h$ for $i \in \{0, \dots, N\}$ and $h = \frac{T}{N}$. Then the exponential Euler scheme reads as follows

$$\begin{aligned} Y_0^N &= \xi \\ Y_{t_{n+1}}^N &= e^{hA}\left[Y_{t_n}^N + F(Y_{t_n}^N) \cdot h + \int_{t_n}^{t_{n+1}} B(Y_{t_n}^N) \,\mathrm{d}W_s \right], \quad n \in \{0, \dots, N-1\}\,. \end{aligned} \tag{1.21}$$

Assuming $F \in \mathrm{Lip}^4(H, H_{-\gamma})$ and $B \in \mathrm{Lip}^4(H, \mathrm{L}_2(U, H_{-\gamma/2}))$ for $\gamma \in [0, \frac{1}{2}]$ this scheme converges with strong order almost $\frac{1-\gamma}{2}$, i.e. for every $\varepsilon \in (0, \infty)$ there exists a constant $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have

$$\left\| X_T - Y_T^N \right\|_{L^2(\Omega, H)} \leq C \cdot N^{-(\frac{1-\gamma}{2} - \varepsilon)}\,. \tag{1.22}$$

Under the same assumptions it was shown in [JK21] that the exponential Euler scheme achieves a weak convergence rate of almost $1 - \gamma$, i.e. for every $\varphi \in \mathrm{Lip}^4(H, V)$ and $\varepsilon \in (0, \infty)$ there exists a constant $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have

$$\left\| \mathbb{E}\left[\varphi(X_T) - \varphi(Y_T^N) \right] \right\|_V \leq C \cdot N^{-(1-\gamma-\varepsilon)}\,. \tag{1.23}$$

This confirms the pattern from finite-dimensional SDEs that “usually” the weak order is twice the strong order.

1.3.1 The Exponential Milstein Scheme

To derive the exponential Milstein scheme we use instead the higher order approximation (1.20) for the integrand of the stochastic integral and then once again (1.19) for the integrand of the resulting iterated stochastic integral. This results in

$$\begin{aligned} X_t &\approx e^{(t-s)A} X_s + \int_s^t e^{(t-s)A} F(e^{(r-s)A} X_s) \,\mathrm{d}r + \int_s^t e^{(t-s)A} B(e^{(r-s)A} X_s) \,\mathrm{d}W_r \\ &\quad + \int_s^t \int_s^r e^{(t-u)A} B'(e^{(r-u)A} X_u) e^{(r-u)A} B(X_u) \,\mathrm{d}W_u \,\mathrm{d}W_r \\ &\approx e^{(t-s)A} X_s + \int_s^t e^{(t-s)A} F(e^{(r-s)A} X_s) \,\mathrm{d}r + \int_s^t e^{(t-s)A} B(e^{(r-s)A} X_s) \,\mathrm{d}W_r \\ &\quad + \int_s^t \int_s^r e^{(t-s)A} B'(e^{(r-s)A} X_s) e^{(r-s)A} B(e^{(u-s)A} X_s) \,\mathrm{d}W_u \,\mathrm{d}W_r \end{aligned}$$

for $(s,t) \in \Lambda$.

Now we again want to use $e^{(t-s)A} \approx \mathrm{Id}_H$ but this time we have to be careful, since in general we cannot compose $B'(x)$ and $B(x)$. Thus, we have to assume $\beta = 0$, i.e. $B \in \mathrm{Lip}(H, \mathrm{L}_2(U, H))$. Then we obtain for $(s,t) \in \Lambda$

$$\begin{aligned}
X_t &\approx e^{(t-s)A}X_s + \int_s^t e^{(t-s)A}F(X_s)\,\mathrm{d}r + \int_s^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_r \\
&\quad + \int_s^t\int_s^r e^{(t-s)A}B'(X_s)B(X_s)\,\mathrm{d}W_u\,\mathrm{d}W_r \\
&\approx e^{(t-s)A}\left[X_s + F(X_s)\cdot(t-s) + \int_s^t B(X_s)\,\mathrm{d}W_s + \int_s^t\int_s^r B'(X_s)B(X_s)\,\mathrm{d}W_u\,\mathrm{d}W_r\right].
\end{aligned}$$

The time-discrete exponential Milstein scheme now reads

$$\begin{aligned}
Z_0^N &= \xi \\
Z_{t_{n+1}}^N &= e^{hA}\Big[Z_{t_n}^N + F(Z_{t_n}^N)\cdot h + \int_{t_n}^{t_{n+1}} B(Z_{t_n}^N)\,\mathrm{d}W_s \\
&\qquad + \int_{t_n}^{t_{n+1}}\int_{t_n}^{r} B'(Z_{t_n}^N)B(Z_{t_n}^N)\,\mathrm{d}W_u\,\mathrm{d}W_r\Big], \quad n\in\{0,\dots,N-1\}.
\end{aligned} \tag{1.24}$$

Assume that $F\in\mathrm{Lip}^2(H,H)$, $B\in\mathrm{Lip}^2(H,\mathrm{L}_2(U,H))$, that there exists $\delta\in(0,\frac{1}{2})$ such that $B(H_\delta)\subset\mathrm{L}_2(U,H_\delta)$ and that the operator $B'B\colon H\to\mathrm{L}_2(\overline{U\otimes U},H)$ is globally Lipschitz. Then it was shown in [JR15] that for regular enough initial values the above scheme converges with strong order almost $\frac{1}{2}+\delta$, i.e. for every $\varepsilon\in(0,\infty)$ there exists a constant $C\in\mathbb{R}$ such that for all $N\in\mathbb{N}$ we have

$$\left\|X_T - Z_T^N\right\|_{L^2(\Omega,H)} \le C\cdot N^{-(\frac{1}{2}+\delta-\varepsilon)}. \tag{1.25}$$

In some cases it is thus possible to achieve a strong order of almost 1. Contrast this with the maximum possible strong order of $\frac{1}{2}$ for the Euler scheme.

In chapter 2, we will show a corresponding estimate for the weak order of the exponential Milstein scheme.

1.3.2 Numerical Approximations as Mild Itô Processes

Now we will see how we can write numerical approximation processes such as (1.21) and (1.24) as a mild Itô process. Fix again an equidistant grid $\{0=t_0,\dots,t_N=T\}$ with step size $h=\frac{T}{N}$. For $t\in[0,T]$ let $\lfloor t\rfloor_h := \lfloor\frac{t}{h}\rfloor\cdot h$ denote the last grid point before or equal to t. Then we define the continuous interpolation of the Euler scheme $(Y_{t_n})_{n\in\{0,\dots,N\}}$ for $t\in[0,T]$ as

$$Y_t^N = e^{tA}\xi + \int_0^t e^{(t-\lfloor s\rfloor_h)A}F(Y_{\lfloor s\rfloor_h}^N)\,\mathrm{d}s + \int_0^t e^{(t-\lfloor s\rfloor_h)A}B(Y_{\lfloor s\rfloor_h}^N)\,\mathrm{d}W_s. \tag{1.26}$$

It is easily seen, that Y does indeed interpolate the values given by (1.21) and defines a mild Itô process with evolution family $S_{s,t}=e^{(t-s)A}$, mild drift $\varphi_s = e^{(s-\lfloor s\rfloor_h)A}F(Y_{\lfloor s\rfloor_h}^N)$ and mild diffusion $\Phi_s = e^{(s-\lfloor s\rfloor_h)A}B(Y_{\lfloor s\rfloor_h}^N)$ for $s\in[0,T]$ and $t\in(s,T]$.

Note how the mild drift and diffusion operators contain a correction term compared to the drift and diffusion of the exact solution. This motivates studying processes of the following form as introduced in [JK21]

$$Y_t^N = S_{0,t}\xi + \int_0^t S_{s,t}R_sF(Y_{\lfloor s\rfloor_h}^N)\,\mathrm{d}s + \int_0^t S_{s,t}R_sB(Y_{\lfloor s\rfloor_h}^N)\,\mathrm{d}W_s, \quad t\in[0,T] \tag{1.27}$$

with a general evolution family $S \in \mathbb{M}(\mathcal{B}(\Lambda), \mathcal{B}(L(H_{-1}, H_{-1}))$ and a numerical correction term $R \in \mathbb{M}(\mathcal{B}([0,T]), \mathcal{B}(L(H_{-1}, H_{-1}))$. We will call the pair (S, R) a *numerical evolution family.* This pair characterises the different "Euler-type" and "Milstein-type" schemes studied in [JK21] and in this work. In order for the numerical evolution family to yield a meaningful variant of the base scheme, it has to be in some way adapted to or consistent with the semigroup generated by A. These consistency requirements are given in the following assumption.

Assumption 1 (Numerical evolution family)**.**
For a step size $h \in (0, \infty)$, time points $(s,t) \in \Lambda$, real numbers $r \in [0,1)$, $\rho \in [0,1)$ and $\tilde{r} \in [-1, 1-r)$ we assume that there exist constants $C_r \in [1,\infty)$, $C_{r,\rho} \in [1,\infty)$ and $C_{r,\tilde{r},\rho} \in [1,\infty)$ such that

$$\begin{aligned} &S_{s,t}(H) \subseteq H, && R_t(H) \subseteq (H), && S_{s,t}R_s(H_{-r}) \subseteq H_{\tilde{r}}, \\ &\|S_{s,t}\|_{L(H,H)} \leq C_0, && \|R_t\|_{L(H,H)} < \infty, && \|S_{s,t}R_s\|_{L(H_{-r},H)} \leq C_r(t-s)^{-r}\,. \end{aligned}$$

We also require that the constants are chosen such that

$$\|e^{tA}\|_{L(H,H_\rho)} \leq C_\rho t^{-\rho}$$

as well as

$$\|e^{tA} - \mathrm{Id}_H\|_{L(H,H_{-\rho})} \leq C_\rho t^{\rho}$$

and finally, that the evolution family is somehow close to the semigroup

$$\|S_{s,t} - e^{(t-s)A}\|_{L(H,H_{-r})} \leq C_{r,\rho} h^\rho (t-s)^{-(\rho-r)^+}$$

and

$$\|S_{s,t}R_s - e^{(t-s)A}\|_{L(H_{-r},H_{\tilde{r}})} \leq C_{r,\tilde{r},\rho} h^\rho (t-s)^{-(\rho+r+\tilde{r})^+}\,.$$

This has the advantage that we can study different types of approximation processes in the same setting. In particular, for the exponential Euler scheme with step size h we have

$$S_{s,t} = e^{(t-s)A}, \quad (s,t) \in \Lambda \tag{1.28}$$

$$R_t = e^{(t-\lfloor t \rfloor_h)A}, \quad t \in [0,T] \tag{1.29}$$

and the above assumptions are fulfilled because we assumed that A is a densely defined and sectorial operator and as such generates an analytic semigroup (see [RR04, Lemma 12.36]). The linear-implicit Euler scheme with step size h is defined by

$$S_{s,t} = (\mathrm{Id}_H - (s - \lfloor s \rfloor_h)A)(\mathrm{Id}_H - (t - \lfloor t \rfloor_h)A)^{-1}(\mathrm{Id}_H - hA)^{-(\lfloor t \rfloor_h - \lfloor s \rfloor_h)/h}, \quad (s,t) \in \Lambda \tag{1.30}$$

$$R_t = (\mathrm{Id}_H - (t - \lfloor t \rfloor_h)A)^{-1}, \quad t \in [0,T] \tag{1.31}$$

and it was shown in [JK21, Proposition 1.3] that this also fulfils the above assumptions.

To deal with Milstein-type approximations we now introduce mild Itô processes of the form

$$Z_t^N = S_{0,t}\xi + \int_0^t S_{s,t}R_sF(Z^N_{\lfloor s\rfloor_h})\,\mathrm{d}s + \int_0^t S_{s,t}R_sG(Z^N_{\lfloor s\rfloor_h}, V_{\lfloor s\rfloor_h,s})\,\mathrm{d}W_s, \quad t\in[0,T] \quad (1.32)$$

where $V\colon \Lambda \to H, (s,t)\mapsto \int_s^t B(Z_s^N)\,\mathrm{d}W_u$ is the stochastic integral of B, $G\colon H\times H \to \mathrm{L}_2(U,H)$ and the evolution family and the correction term still have to satisfy Assumption 1.

Assumption 2 (Approximate diffusion).
The function G has to satisfy the following assumptions:

i) $G\in C^3(H\times H, \mathrm{L}_2(U,H))$,

ii) for all $x\in H$ we have $G(x,0)=B(x)$,

iii) there exists a constant $C_G\in[1,\infty)$ such that for all $x,y\in H$ and $0\le i,j\le 3$ with $i+j\le 3$ we have

$$\Big\|\Big(\frac{\partial^{i+j}}{\partial x^i\,\partial y^j}G\Big)(x,y)\Big\|_{L^{(i+j)}(H^{i+j},\mathrm{L}_2(U,H))} \le C_G\max\{1,\|x\|_H,\|y\|_H\}$$

iv) and for every $p\ge 1$, all $0\le s<t$ and $\mathcal{F}_s$-measurable $X,Y\in L^p(\Omega,H)$ we have

$$\begin{aligned}\Big\|G(X,\int_s^t B(X)\,\mathrm{d}W_u) - G(Y,\int_s^t B(Y)\,\mathrm{d}W_u)\Big\|_{L^p(\Omega,\mathrm{L}_2(U,H))}\\ \le C_G(1+\tau_p(t-s)^{\frac{1}{2}})\|X-Y\|_{L^p(\Omega,H)}.\end{aligned}$$

We can choose $G(x,y)=B(x)+B'(x)y$ to obtain the exponential Milstein scheme (1.24). We will now show that this choice of G indeed fulfils the assumptions.

Lemma 1.39.
Let $B\in\mathrm{Lip}^4(H,\mathrm{L}_2(U,H))$ such that $B'B\colon H\to\mathrm{L}_2(\overline{U\otimes U},H)$ is globally Lipschitz, i.e. there exists a constant $|B'B|_{\mathrm{Lip}(H,\mathrm{L}_2(\overline{U\otimes U},H))}\in[0,\infty)$ such that for all $x,y\in H$ it holds that

$$\|B'(x)B(x)-B'(y)B(y)\|_{\mathrm{L}_2(\overline{U\otimes U},H)} \le |B'B|_{\mathrm{Lip}(H,\mathrm{L}_2(\overline{U\otimes U},H))}\cdot\|x-y\|_H.$$

Then the function $G(x,y)=B(x)+B'(x)y$ fulfils Assumption 2.

Proof. The first two assumptions are clearly fulfilled. Furthermore, it holds that

$$\begin{aligned}\|G(x,y)\|_{\mathrm{L}_2(U,H)} &\le \|B(0)\|_{\mathrm{L}_2(U,H)} + |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\|x\|_H + |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\|y\|_H\\ &\le 2\|B\|_{\mathrm{Lip}^4(H,\mathrm{L}_2(U,H))}\cdot\max\{1,\|x\|_H,\|y\|_H\}\end{aligned}$$

and similarly for the derivatives of G, such that we have at least $C_G\ge 2\|B\|_{\mathrm{Lip}^4(H,\mathrm{L}_2(U,H))}$.

Checking the last assumption let $p \geq 1$, $0 \leq s < t$ and $X, Y \in L^p(\Omega, H)$ be $\mathcal{F}_s$-measurable. Then we have

$$
\begin{aligned}
&\Big\| G\big(X, \int_s^t B(X)\,\mathrm{d}W_u\big) - G\big(Y, \int_s^t B(Y)\,\mathrm{d}W_u\big) \Big\|_{L^p(\Omega, \mathrm{L}_2(U,H))} \\
&\qquad \leq \|B(X) - B(Y)\|_{L^p(\Omega, \mathrm{L}_2(U,H))} \\
&\qquad\qquad + \Big\| \int_s^t B'(X)B(X) - B'(Y)B(Y)\,\mathrm{d}W_u \Big\|_{L^p(\Omega, \mathrm{L}_2(U,H))} \\
&\qquad \leq |B|_{\mathrm{Lip}(H, \mathrm{L}_2(U,H))} \|X - Y\|_{L^p(\Omega, H)} \\
&\qquad\qquad + \tau_p \left(\int_s^t \|B'(X)B(X) - B'(Y)B(Y)\|^2_{L^p(\Omega, \mathrm{L}_2(\overline{U \otimes U}, H))}\,\mathrm{d}u \right)^{\frac{1}{2}} \\
&\qquad \leq \max\big\{ |B|_{\mathrm{Lip}(H, \mathrm{L}_2(U,H))}, |B'B|_{\mathrm{Lip}(H, \mathrm{L}_2(\overline{U \otimes U}, H))} \big\} (1 + \tau_p (t-s)^{\frac{1}{2}}) \|X - Y\|_{L^p(\Omega, H)}
\end{aligned}
$$

which means that we must have $C_G \geq \max\big\{ |B|_{\mathrm{Lip}(H, \mathrm{L}_2(U,H))}, |B'B|_{\mathrm{Lip}(H, \mathrm{L}_2(\overline{U \otimes U}, H))} \big\}$.

Overall we can choose $C_G = \max\big\{ 2\|B\|_{\mathrm{Lip}^4(H, \mathrm{L}_2(U,H))}, |B'B|_{\mathrm{Lip}(H, \mathrm{L}_2(\overline{U \otimes U}, H))} \big\}$. □

In the next chapters we will show that the processes (1.32) converge with a weak convergence rate of almost $1 - \alpha$ to the true mild solution of (SEE).

Chapter 2

Weak Convergence for Regular Coefficients

In this chapter we establish the weak convergence rates first for suitably regularised versions of the processes. In particular, for the main result in this chapter in section 2.5 we will assume that the coefficients of the equation take values in the small space $H_2 \subseteq H$. This is a serious limitation and will be overcome in chapter 3, where we show how to transfer the result to the general setting. As some of the results do not require these strong regularity assumptions, we will first work in the general setting detailed below and only in the final section come back and impose these stronger assumptions.

We now detail the general setting. Let $T > 0$ be a given final time. Let $(A, \mathrm{D}(A))$ be a densely defined sectorial operator on the separable Hilbert space H with angle bigger than $\pi/2$. Without loss of generality (see Remark 1.26) we also assume that $0 \in \rho(A)$. By Theorem 1.17 we then see that A generates a bounded analytic semigroup $(e^{tA})_{t\in[0,T]}$ and associated interpolation spaces $(H_\alpha)_{\alpha\in\mathbb{R}}$ such that the semigroup exhibits the smoothing properties of Theorem 1.24. The coefficients of the stochastic evolution equation have to satisfy the following assumptions.

Assumption 3.
Let $p \geq 2$ and $\vartheta \in [0, 1)$. We assume that the initial value satisfies $\xi \in L^p(\Omega, H)$, that the drift coefficient satisfies $F \in \mathrm{Lip}^4(H, H_{-\vartheta})$ and that the diffusion coefficient satisfies $B \in \mathrm{Lip}^4(H, \mathrm{L}_2(U, H))$.

These assumptions guarantee by Theorem 1.28 the existence of a unique mild solution to the SEE

$$\begin{cases} \mathrm{d}X_t = [AX_t + F(X_t)]\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t, & t \in (0, T], \\ X_0 = \xi. \end{cases}$$

Furthermore, we will need a numerical evolution family (S, R) satisfying Assumption 1 to define a specific variant of the Milstein scheme. Finally, we always work on an equidistant grid $\{0 = t_0, \dots, t_N = T\}$ with step size $h = \frac{T}{N}$. Remember that for $t \in [0, T]$ we denote by $\lfloor t \rfloor_h$ the last grid point before or equal to t.

In this setting we will study the true mild solution $X \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$, its numerical Milstein-type approximation $Z \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$ as well as a suitably chosen "intermediate" process $\bar{Z} \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$. These processes are defined by $X_0 = Z_0 = \bar{Z}_0 = \xi$ and

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A} B(X_s)\,\mathrm{d}W_s, \tag{2.1}$$

$$Z_t = S_{0,t}\xi + \int_0^t S_{s,t}R_s F(Z_{\lfloor s \rfloor_h})\,\mathrm{d}s + \int_0^t S_{s,t}R_s G\Big(Z_{\lfloor s \rfloor_h}, \int_{\lfloor s \rfloor_h}^s B(Z_{\lfloor s \rfloor_h})\,\mathrm{d}W_u\Big)\,\mathrm{d}W_s \tag{2.2}$$

and

$$\bar{Z}_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(Z_{\lfloor s \rfloor_h})\,\mathrm{d}s + \int_0^t e^{(t-s)A} G\Big(Z_{\lfloor s \rfloor_h}, \int_{\lfloor s \rfloor_h}^s B(Z_{\lfloor s \rfloor_h})\,\mathrm{d}W_u\Big)\,\mathrm{d}W_s \tag{2.3}$$

for $t \in (0,T]$. Both approximating processes implicitly depend on the current step size of the time discretisation.

In Lemma 2.8 we will first show that $\bar{Z}$ and X are close in the weak topology. This is the central proof of this work and represents the main difficulty. Here we need the earlier facts about the regularity of the Kolmogorov equation as well as the theory of mild Itô processes. As a byproduct we also obtain the convergence of Z to $\bar{Z}$ in Lemma 2.9. Combining these with an elementary application of the triangle inequality in Corollary 2.10 we prove the weak convergence of Z to X.

This chapter is structured as follows. In section 2.1, we show that the approximating processes (which start in $\xi \in L^p(\Omega, H)$) stay bounded in L^p up until the final time T. Next, in section 2.2 we establish a kind of strong convergence result between Z and $\bar{Z}$ in the weak $L^p(\Omega, H_{-\rho})$ space. This is used in section 2.3 to show two weak convergence results for locally Lipschitz test functions with a specific form. After showing in section 2.4 that the terms on the right-hand side of the Kolmogorov backward equation satisfy these local Lipschitz bounds we can apply the previous results in section 2.5 to finally prove the main weak convergence result.

2.1 Moment Bounds

First we need to establish suitable bounds on the moments of our two approximating processes Z and $\bar{Z}$. In this chapter it will be enough to know that the moments stay finite on $[0,T]$. Later we will strengthen this result in section 3.2.1 as we will need explicit estimates to carry out the mollification. Define for $r \in (0,\infty)$ the constant

$$K_r = \sup_{s,t\in[0,T]} \Big\| \max\{1, \|Z_s\|_H, \|\bar{Z}_t\|_H\} \Big\|_{L^r(\Omega)} \in \overline{\mathbb{R}}. \tag{2.4}$$

This constant provides an upper bound for the moments in the sense that $\|Z_t\|_{L^r(\Omega,H)} \leq K_r$ for all $t \in [0,T]$ and $r \in (0,\infty)$ and similarly for $\bar{Z}$. In the next lemma we collect some calculations that we will need frequently. Remember that we denote the stochastic integral of B by $V_{s,t} = \int_s^t B(Z_s)\,\mathrm{d}W_u$.

Lemma 2.1.

i) For all $(s,t) \in \Lambda$ and $r \in (0,p]$ there exists a constant $C_{V,t-s,r} \in [0,\infty)$ such that

$$\|V_{s,t}\|_{L^r(\Omega,H)} \leq C_{V,t-s,r} \cdot \|\max\{1, \|Z_s\|_H\}\|_{L^r(\Omega)}.$$

We can choose $C_{V,t-s,r} = \tau_r \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}(t-s)^{\frac{1}{2}}$*. In particular, we have*

$$\sup_{s\in[0,T]} \sup_{r\in[2,p]} C_{V,s,r} = C_{V,T,p}.$$

ii) *For all* $(s,t) \in \Lambda$ *and* $r \in [1,p]$ *we have*

$$\|G(Z_s, V_{s,t})\|_{L^r(\Omega,\mathrm{L}_2(U,H))} \leq C_G(1 + C_{V,t-s,r})\|\max\{1, \|Z_s\|_H\}\|_{L^r(\Omega)}.$$

iii) *For all* $(s,t) \in \Lambda$*,* $u, v \in [0,T]$ *and* $r \in [1,p]$ *it holds that*

$$\|\max\{1, \|Z_u\|_H, \|\bar{Z}_v\|_H, \|V_{s,t}\|_H\}\|_{L^r(\Omega)} \leq (1 + C_{V,t-s,r})K_r.$$

Proof.

i) The inequality follows from Corollary 1.10:

$$\begin{aligned}\|V_{s,t}\|_{L^r(\Omega,H)} &\leq \tau_r \mathbb{E}\left[\left(\int_s^t \|B(Z_s)\|^2_{\mathrm{L}_2(U,H)}\,\mathrm{d}u\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} \\ &\leq \tau_r \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}(t-s)^{\frac{1}{2}}\|\max\{1, \|Z_s\|_H\}\|_{L^r(\Omega)}.\end{aligned}$$

Monotonicity of the constant follows from the monotonicity of τ_r for $r \geq 2$.

ii) This follows from Assumption 2 and i):

$$\begin{aligned}\|G(Z_s, V_{s,t})\|_{L^r(\Omega,\mathrm{L}_2(U,H))} &\leq \|C_G \max\{1, \|Z_s\|_H, \|V_{s,t}\|_H\}\|_{L^r(\Omega)} \\ &\leq C_G(\|\max\{1, \|Z_s\|_H\}\|_{L^r(\Omega)} + \|V_{s,t}\|_{L^r(\Omega,H)}) \\ &\leq C_G(1 + C_{V,t-s,r})\|\max\{1, \|Z_s\|_H\}\|_{L^r(\Omega)}.\end{aligned}$$

iii) Similar to the previous one we have

$$\begin{aligned}&\|\max\{1, \|Z_u\|_H, \|\bar{Z}_v\|_H, \|V_{s,t}\|_H\}\|_{L^r(\Omega)} \\ &\quad\leq \|\max\{1, \|Z_u\|_H, \|\bar{Z}_v\|_H\}\|_{L^r(\Omega)} + \|V_{s,t}\|_{L^r(\Omega,H)} \\ &\quad\leq (1 + C_{V,t-s,r})K_r.\end{aligned}$$

□

Lemma 2.2.
Let $\vartheta \in [0,1)$*. It holds that*

$$\sup_{r\in(0,p]} K_r = K_p < \infty.$$

Proof. The claimed equality follows readily from Jensen's inequality. To show that K_p is finite, we first show that the pth moment of Z is finite at the grid points. We then deduce that also the moments at the intermediate time points have to be finite. This is in essence a straightforward consequence of the used interpolation scheme. We make repeated use of the Burkholder–Davis–Gundy inequality in the form of Corollary 1.10.

For $n \in \{1, \dots, N\}$ it holds with Corollary 1.10:

$$\begin{aligned}
\|Z_{t_n}\|_{L^p(\Omega,H)} &\le \|S_{0,t_n}Z_0\|_{L^p(\Omega,H)} + \left\|\int_0^{t_n} S_{s,t_n}R_sF(Z_{\lfloor s\rfloor_h})\,\mathrm{d}s\right\|_{L^p(\Omega,H)} \\
&\quad + \left\|\int_0^{t_n} S_{s,t_n}R_sG(Z_{\lfloor s\rfloor_h}, V_{\lfloor s\rfloor_h,s})\,\mathrm{d}W_s\right\|_{L^p(\Omega,H)}
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&\le C_0\|Z_0\|_{L^p(\Omega,H)} + \int_0^{t_n} \frac{C_\vartheta}{(t_n-s)^\vartheta}\|F(Z_{\lfloor s\rfloor_h})\|_{L^p(\Omega,H_{-\vartheta})}\,\mathrm{d}s \\
&\quad + C_0\tau_p\left(\int_0^{t_n}\left\|G(Z_{\lfloor s\rfloor_h}, V_{\lfloor s\rfloor_h,s})\right\|^2_{L^p(\Omega,\mathrm{L}_2(U,H))}\,\mathrm{d}s\right)^{1/2}
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\le C_0\|Z_0\|_{L^p(\Omega,H)} + \int_0^{t_n} \frac{C_\vartheta\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{(t_n-s)^\vartheta}\left\|\max\{1,\|Z_{\lfloor s\rfloor_h}\|_H\}\right\|_{L^p(\Omega)}\,\mathrm{d}s \\
&\quad + C_0\tau_p\left(\int_0^{t_n} C_G^2(1+C_{V,s-\lfloor s\rfloor_h,p})^2\left\|\max\{1,\|Z_{\lfloor s\rfloor_h}\|_H\}\right\|^2_{L^p(\Omega)}\,\mathrm{d}s\right)^{1/2}
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\le \left[C_0 + C_\vartheta\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\frac{t_n^{1-\vartheta}}{1-\vartheta} + \tau_pC_0C_G(1+C_{V,h,p})t_n^{\frac{1}{2}}\right] \\
&\quad \cdot \max_{j\in\{0,\dots,n-1\}}\left\|\max\{1,\|Z_{t_j}\|_H\}\right\|_{L^p(\Omega)}
\end{aligned} \tag{2.8}$$

By induction we thus get

$$\sup_{t\in[0,T]}\left\|Z_{\lfloor t\rfloor_h}\right\|_{L^p(\Omega,H)} = \max_{n\in\{0,\dots,N\}}\|Z_{t_n}\|_{L^p(\Omega,H)} < \infty \tag{2.9}$$

which is finite since we assumed $Z_0 = \xi \in L^p(\Omega,H)$. Using again Corollary 1.10, we can then see that for all $t \in [0,T]$ we have

$$\begin{aligned}
\|Z_t\|_{L^p(\Omega,H)} &\le \|S_{0,t}Z_0\|_{L^p(\Omega,H)} + \int_0^t \|S_{s,t}R_sF(Z_{\lfloor s\rfloor_h})\|_{L^p(\Omega,H)}\,\mathrm{d}s \\
&\quad + \tau_p\left(\int_0^t \|S_{s,t}R_sG(Z_{\lfloor s\rfloor_h}, V_{\lfloor s\rfloor_h,s})\|^2_{L^p(\Omega,\mathrm{L}_2(U,H))}\,\mathrm{d}s\right)^{\frac{1}{2}}
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\le C_0\|Z_0\|_{L^p(\Omega,H)} + \int_0^t \frac{C_\vartheta}{(t-s)^\vartheta}\|F(Z_{\lfloor s\rfloor_h})\|_{L^p(\Omega,H_{-\vartheta})}\,\mathrm{d}s \\
&\quad + \tau_pC_0\left(\int_0^t C_G^2(1+C_{V,s-\lfloor s\rfloor_h,p})^2\left\|\max\{1,\|Z_{\lfloor s\rfloor_h}\|_H\}\right\|^2_{L^p(\Omega)}\,\mathrm{d}s\right)^{1/2}
\end{aligned} \tag{2.11}$$

$$\leq \left[C_0 + C_\vartheta \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \frac{T^{1-\vartheta}}{1-\vartheta} + \tau_p C_0 C_G (1 + C_{V,T,p}) T^{\frac{1}{2}}\right] \cdot \sup_{s\in[0,T]} \left\| \max\{1, \|Z_{\lfloor s \rfloor_h}\|_H\} \right\|_{L^p(\Omega)} \tag{2.12}$$

which together with (2.9) and the fact that $\sup_{s\in[0,T]} \|\max\{1, \|Z_{\lfloor s \rfloor_h}\|_H\}\|_{L^r(\Omega)} \leq 1 + \sup_{s\in[0,T]} \|Z_{\lfloor s \rfloor_h}\|_{L^r(\Omega,H)}$ shows that

$$\sup_{t\in[0,T]} \|Z_t\|_{L^p(\Omega,H)} < \infty. \tag{2.13}$$

Now we still have to deal with the auxiliary process $\bar{Z}$. Using one last time Corollary 1.10 we see that

$$\sup_{t\in[0,T]} \left\|\bar{Z}_t\right\|_{L^p(\Omega,H)} \leq \left[C_0 + C_\vartheta \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \frac{T^{1-\vartheta}}{1-\vartheta} + \tau_p C_0 C_G (1 + C_{V,h,p}) T^{\frac{1}{2}}\right] \cdot \sup_{t\in[0,T]} \|\max\{1, \|Z_t\|_H\}\|_{L^p(\Omega)} . \tag{2.14}$$

Combining this with (2.13) we conclude

$$K_p = \sup_{s,t\in[0,T]} \left\| \max\{1, \|Z_s\|_H, \|\bar{Z}_t\|_H\} \right\|_{L^p(\Omega)} \tag{2.15}$$

$$\leq \sup_{t\in[0,T]} \|\max\{1, \|Z_t\|_H\}\|_{L^p(\Omega)} + \sup_{t\in[0,T]} \left\|\bar{Z}_t\right\|_{L^p(\Omega,H)} \tag{2.16}$$

$$\leq 2\left[C_0 + C_\vartheta \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \frac{T^{1-\vartheta}}{1-\vartheta} + \tau_p C_0 C_G (1 + C_{V,h,p}) T^{\frac{1}{2}}\right] \cdot \sup_{t\in[0,T]} \|\max\{1, \|Z_t\|_H\}\|_{L^p(\Omega)} < \infty. \tag{2.17}$$

□

2.2 Strong Distance Between the Approximation and the Auxiliary Process

Lemma 2.3.
Let $\vartheta \in [0,1)$, $r \in [1,p]$ and $\rho \in [0,1)$. Then we have

$$\sup_{t\in[0,T]} \left\|Z_t - \bar{Z}_t\right\|_{L^r(\Omega,H_{-\rho})} \leq A_{T,\rho,r} K_r h^\rho$$

where the constant is given by

$$A_{T,\rho,r} := C_{\rho,\rho} + C_{\vartheta,-\rho,\rho} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \frac{T^{1-\vartheta}}{1-\vartheta} + \tau_r C_{0,-\rho,\rho} C_G (1 + C_{V,h,r}) T^{\frac{1}{2}} \tag{2.18}$$

$$\leq 2(C_{\rho,\rho} + C_{\vartheta,-\rho,\rho} + \tau_r C_{0,-\rho,\rho}) \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta} + \frac{T}{1-\rho}\right) \cdot \max\{1, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\} \cdot \max\{1, C_G(1 + C_{V,h,r})\}. \tag{2.19}$$

Proof. First we use the triangle inequality and then estimate each term separately using Assumption 1 on the evolution family. Fix $t \in [0,T]$.

$$\begin{aligned}\left\|Z_t - \bar{Z}_t\right\|_{L^p(\Omega,H_{-\rho})} \leq{}& \left\|\left(S_{0,t} - e^{tA}\right) Z_0\right\|_{L^r(\Omega,H_{-\rho})} \\ &+ \left\|\int_0^t \left(S_{s,t}R_s - e^{(t-s)A}\right) F(Z_{\lfloor s\rfloor_h})\,\mathrm{d}s\right\|_{L^r(\Omega,H_{-\rho})} \\ &+ \left\|\int_0^t \left(S_{s,t}R_s - e^{(t-s)A}\right) G(Z_{\lfloor s\rfloor_h}, V_{\lfloor s\rfloor_h,s})\,\mathrm{d}W_s\right\|_{L^r(\Omega,H_{-\rho})}.\end{aligned} \tag{2.20}$$

For the first term we get

$$\begin{aligned}\left\|\left(S_{0,t} - e^{tA}\right) Z_0\right\|_{L^r(\Omega,H_{-\rho})} &\leq \left\|S_{0,t} - e^{tA}\right\|_{L(H,H_{-\rho})}\|Z_0\|_{L^r(\Omega,H)} \\ &\leq C_{\rho,\rho}h^\rho\|Z_0\|_{L^r(\Omega,H)} \\ &\leq C_{\rho,\rho}K_r h^\rho.\end{aligned} \tag{2.21}$$

For the second term we have

$$\begin{aligned}&\left\|\int_0^t \left(S_{s,t}R_s - e^{(t-s)A}\right) F(Z_{\lfloor s\rfloor_h})\,\mathrm{d}s\right\|_{L^r(\Omega,H_{-\rho})} \\ &\quad\leq \int_0^t \left\|S_{s,t}R_s - e^{(t-s)A}\right\|_{L(H_{-\vartheta},H_{-\rho})}\left\|F(Z_{\lfloor s\rfloor_h})\right\|_{L^r(\Omega,H_{-\vartheta})}\,\mathrm{d}s \\ &\quad\leq \int_0^t C_{\vartheta,-\rho,\rho}h^\rho(t-s)^{-\vartheta}\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\left\|\max\{1,\|Z_{\lfloor s\rfloor_h}\|_H\}\right\|_{L^r(\Omega)}\,\mathrm{d}s \\ &\quad\leq C_{\vartheta,-\rho,\rho}\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\frac{t^{1-\vartheta}}{1-\vartheta}K_r h^\rho.\end{aligned} \tag{2.22}$$

Here we need $\vartheta < 1$ as well as $\rho < 1$. Lastly, we use Corollary 1.10 to deal with the third term

$$\begin{aligned}&\left\|\int_0^t \left(S_{s,t}R_s - e^{(t-s)A}\right) G(Z_{\lfloor s\rfloor_h}, V_{\lfloor s\rfloor_h,s})\,\mathrm{d}W_s\right\|_{L^r(\Omega,H_{-\rho})} \\ &\quad\leq \tau_r\left(\int_0^t \left\|S_{s,t}R_s - e^{(t-s)A}\right\|^2_{L(H,H_{-\rho})}\left\|G(Z_{\lfloor s\rfloor_h}, V_{\lfloor s\rfloor_h,s})\right\|^2_{L^r(\Omega,\mathrm{L}_2(U,H))}\,\mathrm{d}s\right)^{1/2} \\ &\quad\leq \tau_r\left(\int_0^t C^2_{0,-\rho,\rho}h^{2\rho}C_G^2(1+C_{V,s-\lfloor s\rfloor_h,r})^2\left\|\max\{1,\|Z_{\lfloor s\rfloor_h}\|_H\}\right\|^2_{L^r(\Omega)}\,\mathrm{d}s\right)^{1/2} \\ &\quad\leq \tau_r C_{0,-\rho,\rho}C_G(1+C_{V,h,r})t^{\frac{1}{2}}K_r h^\rho.\end{aligned} \tag{2.23}$$

Taking the supremum over all $t \in [0,T]$ gives the claimed estimate. □

2.3 Weak Distance Between the Approximation and the Auxiliary Process

In this section we prove two estimates that somehow characterise the weak distance between our approximation scheme Z and the auxiliary process $\bar{Z}$. The estimates are phrased in terms of a test function ψ fulfilling some kind of local Lipschitz estimates. As we will use the mild Itô formula in the proof, we need at least two derivatives and then also the local Lipschitz estimates for all second order derivatives. We will then show in the next section that we can choose $\psi(x,y,z) = (\frac{\partial}{\partial x}u)(t,x)F(y)$ and $\psi(x,y,z) = \sum_{n\in\mathbb{N}} (\frac{\partial^2}{\partial x^2}u)(t,x)\,(G(y,z)e_n, G(y,z)e_n)$. Therefore, we need at least four derivatives of the solution u. This is guaranteed by Theorem 1.37 as long as the coefficients F and B as well as the test function φ are Lip^4.

The following two propositions are the analogues for our Milstein-type processes of Proposition 5.3 and Proposition 5.5 in [JK21]. The first result deals with a kind of smoothness in time of the auxiliary process. The third parameter in the test function is the stochastic integral of $B(Z_s)$ and vanishes in the second term as we have $V_{s,s} = 0$.

Proposition 2.4.
Let $\vartheta \in [0,1)$, $\rho \in [0, 1-\vartheta)$ and $\psi \in C^2(H\times H\times H, V)$ be given. Assume that there exist $q \in [0, p-3]$ and $\eta \in [0,\infty)$ such that for all $x_1, x_2, y, z \in H$ we have

$$\begin{aligned}\max_{\substack{i,j,k\in\mathbb{N}_0\\ i+j+k\le 2}} &\left\| \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x_1,y,z) - \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x_2,y,z)\right\|_{L^{(i+j+k)}(H,V)} \qquad (2.24)\\ &\le \eta \max\{1, \|x_1\|_H, \|x_2\|_H, \|y\|_H, \|z\|_H\}^q \|x_1 - x_2\|_H.\end{aligned}$$

Then there exists a constant $C\in[0,\infty)$ such that it holds for all $(s,t)\in\Lambda$ that

$$\left\|\mathbb{E}\left[\psi\left(\bar{Z}_t, Z_s, V_{s,t}\right) - \psi\left(\bar{Z}_s, Z_s, 0\right)\right]\right\|_V \le C\cdot\frac{(t-s)^\rho}{t^\rho}. \qquad (2.25)$$

Proof. In the proof we will use Proposition 1.32 repeatedly and then estimate all terms using a combination of the assumption (2.24), the estimates on the evolution family (Assumption 1) and standard tools like the triangle and Hölder inequalities. For the rest of the proof fix $(s,t)\in\Lambda$ and define for $r\in(0,p]$

$$\sigma := \max\{1, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\}$$

and

$$\tilde{\sigma}_r := \max\{1, \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}, C_G(1+C_{V,T,r})\}.$$

For the sake of readability, we will use the subscript notation $\psi_{i,j,k}$ to denote the partial derivatives $\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi$.

Let us start by applying Proposition 1.32 to the mild Itô process $(Y_r)_{r\in[s,t]}$ defined by

$$Y_r = \begin{pmatrix}\bar{Z}_r\\ Z_s\\ V_{s,r}\\ \bar{Z}_s\end{pmatrix},\ \varphi_r = \begin{pmatrix}F(Z_{\lfloor r\rfloor_h})\\ 0\\ 0\\ 0\end{pmatrix},\ \Phi_r = \begin{pmatrix}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})\\ 0\\ B(Z_s)\\ 0\end{pmatrix},\ S^Y_{t_1,t_2} = \begin{pmatrix}e^{(t_2-t_1)A}\\ \mathrm{Id}_H\\ \mathrm{Id}_H\\ \mathrm{Id}_H\end{pmatrix} \qquad (2.26)$$

for $r \in [s,t]$ and $s \le t_1 < t_2 \le t$, $\check{H} = \hat{H} = H^4$ and the function $f\colon \check{H} \to V$, $(w,x,y,z) \mapsto \psi(w,x,y) - \psi(z,x,0)$. The function satisfies the polynomial growth assumption, since we have for all $(w,x,y,z) \in \check{H}$

$$\begin{aligned}
\|f(w,x,y,z)\|_V &= \|\psi(w,x,y) - \psi(z,x,0)\|_V \\
&\le \|\psi(w,x,y) - \psi(z,x,y)\|_V + \|\psi(z,x,y) - \psi(z,x,0)\|_V \\
&\le \eta \max\{1, \|w\|_H, \|x\|_H, \|y\|_H, \|z\|_H\}^q \|w-z\|_H \\
&\qquad + \left\| \int_0^1 \psi_{0,0,1}(z,x,ry)y \,\mathrm{d}r \right\|_H \\
&\le 2\eta \max\{1, \|w\|_H, \|x\|_H, \|y\|_H, \|z\|_H\}^{q+1} \\
&\qquad + \eta \max\{1, \|x\|_H, \|y\|_H, \|z\|_H\}^q \|y\|_H \\
&\le 3\eta \max\{1, \|w\|_H, \|x\|_H, \|y\|_H, \|z\|_H\}^{q+1} \\
&\le 3\eta \left(1 + \left(\|w\|_H^2 + \|x\|_H^2 + \|y\|_H^2 + \|z\|_H^2 \right)^{\frac{q+1}{2}} \right).
\end{aligned}$$

This results in

$$\begin{aligned}
&\left\| \mathbb{E}\Big[\psi\left(\bar{Z}_t, Z_s, V_{s,t}\right) - \psi\left(\bar{Z}_s, Z_s, 0\right) \Big] \right\|_V \le \left\| \mathbb{E}\Big[\psi\left(e^{(t-s)A}\bar{Z}_s, Z_s, 0\right) - \psi\left(\bar{Z}_s, Z_s, 0\right) \Big] \right\|_V \\
&\quad + \int_s^t \mathbb{E}\Big[\Big\| \psi_{1,0,0}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r}) e^{(t-r)A} F(Z_{\lfloor r \rfloor_h}) \Big\|_V \Big] \,\mathrm{d}r \\
&\quad + \frac{1}{2} \int_s^t \mathbb{E}\Big[\Big\| \sum_{n\in\mathbb{N}} \psi_{2,0,0}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})(e^{(t-r)A} G(Z_{\lfloor r \rfloor_h}, V_{\lfloor r \rfloor_h, r}) e_n, \\
&\qquad e^{(t-r)A} G(Z_{\lfloor r \rfloor_h}, V_{\lfloor r \rfloor_h, r}) e_n) \Big\|_V \Big] \,\mathrm{d}r \\
&\quad + \int_s^t \mathbb{E}\Big[\Big\| \sum_{n\in\mathbb{N}} \psi_{1,0,1}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})(e^{(t-r)A} G(Z_{\lfloor r \rfloor_h}, V_{\lfloor r \rfloor_h, r}) e_n, B(Z_s)e_n) \Big\|_V \Big] \,\mathrm{d}r \\
&\quad + \frac{1}{2} \int_s^t \mathbb{E}\Big[\Big\| \sum_{n\in\mathbb{N}} \psi_{0,0,2}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})(B(Z_s)e_n, B(Z_s)e_n) \Big\|_V \Big] \,\mathrm{d}r.
\end{aligned} \tag{2.27}$$

For the first term we will again use Proposition 1.32. The other terms will be handled directly.

We start by estimating the integrands using the assumptions on ψ and the Hölder inequality. To this end, we observe that the assumption (2.24) implies that for all $x, y, z \in H$

$$\|\psi_{1,0,0}(x,y,z)\|_{L(H,V)} = \sup_{\substack{h\in H \\ \|h\|_H = 1}} \|\psi_{1,0,0}(x,y,z)h\|_V \tag{2.28}$$

$$= \sup_{\substack{h\in H \\ \|h\|_H = 1}} \lim_{t\to 0} \frac{\|\psi(x+th,y,z) - \psi(x,y,z)\|_V}{t} \tag{2.29}$$

$$\le \sup_{\substack{h\in H\\ \|h\|_H=1}} \lim_{t\to 0} \frac{\eta \max\{1, \|x+th\|_H, \|x\|_H, \|y\|_H, \|z\|_H\}^q \|th\|_H}{t} \tag{2.30}$$

$$\le \eta \max\{1, \|x\|_H, \|y\|_H, \|z\|_H\}^q \tag{2.31}$$

and analogously for the other integrands. We then get

$$\mathbb{E}\Big[\Big\|\psi_{1,0,0}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})e^{(t-r)A}F(Z_{\lfloor r\rfloor_h})\Big\|_V\Big] \tag{2.32}$$

$$\le \eta\mathbb{E}\Big[\max\{1, \|e^{(t-r)A}\bar{Z}_r\|_H, \|Z_s\|_H, \|V_{s,r}\|_H\}^q \|e^{(t-r)A}F(Z_{\lfloor r\rfloor_h})\|_H\Big] \tag{2.33}$$

$$\begin{aligned}\le \eta C_0^q \|e^{(t-r)A}\|_{L(H_{-\vartheta},H)} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\\ \cdot \Big\|\max\{1, \|\bar{Z}_r\|_H, \|Z_s\|_H, \|V_{s,r}\|_H\}\Big\|_{L^{q+1}(\Omega)}^q \Big\|\max\{1, \|Z_{\lfloor r\rfloor_h}\|_H\}\Big\|_{L^{q+1}(\Omega)}\end{aligned} \tag{2.34}$$

$$\le \eta C_0^q C_\vartheta (t-r)^{-\vartheta} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} (1 + C_{V,T,q+1})^q K_{q+1}^{q+1}, \tag{2.35}$$

$$\begin{aligned}\mathbb{E}\Big[\Big\|\sum_{n\in\mathbb{N}} \psi_{2,0,0}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})\big(e^{(t-r)A}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})e_n,\\ e^{(t-r)A}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})e_n\big)\Big\|\Big]\end{aligned} \tag{2.36}$$

$$\le \eta\mathbb{E}\Big[\max\{1, \|e^{(t-r)A}\bar{Z}_r\|_H, \|Z_s\|_H, \|V_{s,r}\|_H\}^q \sum_{n\in\mathbb{N}} \|e^{(t-r)A}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})e_n\|_H^2\Big] \tag{2.37}$$

$$\le \eta C_0^{q+2} \Big\|\max\{1, \|\bar{Z}_r\|_H, \|Z_s\|_H, \|V_{s,r}\|_H\}\Big\|_{L^{q+2}(\Omega)}^q \Big\|G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})\Big\|_{L^{q+2}(\Omega,\mathrm{L}_2(U,H))}^2 \tag{2.38}$$

$$\le \eta C_0^{q+2} C_G^2 (1 + C_{V,T,q+2})^{q+2} K_{q+2}^{q+2}, \tag{2.39}$$

$$\mathbb{E}\Big[\Big\|\sum_{n\in\mathbb{N}} \psi_{1,0,1}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})\big(e^{(t-r)A}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})e_n, B(Z_s)e_n\big)\Big\|_V\Big] \tag{2.40}$$

$$\begin{aligned}\le \eta\mathbb{E}\Big[\max\{1, \|e^{(t-r)A}\bar{Z}_r\|_H, \|Z_s\|_H, \|V_{s,r}\|_H\}^q\\ \sum_{n\in\mathbb{N}} \|e^{(t-r)A}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})e_n\|_H \|B(Z_s)e_n\|_H\Big]\end{aligned} \tag{2.41}$$

$$\begin{aligned}\le \eta C_0^{q+1} \Big\|\max\{1, \|\bar{Z}_r\|_H, \|Z_s\|_H, \|V_{s,r}\|_H\}\Big\|_{L^{q+2}(\Omega)}^q\\ \Big\|G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})\Big\|_{L^{q+2}(\Omega,\mathrm{L}_2(U,H))} \|B(Z_s)\|_{L^{q+2}(\Omega,\mathrm{L}_2(U,H))}\end{aligned} \tag{2.42}$$

$$\le \eta C_0^{q+1} C_G \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))} (1 + C_{V,T,q+2})^{q+1} K_{q+2}^{q+2}, \tag{2.43}$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{n\in\mathbb{N}} \psi_{0,0,2}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})(B(Z_s)e_n, B(Z_s)e_n)\right\|_V\right] && (2.44)\\
&\qquad\le \eta\mathbb{E}\left[\max\{1, \|e^{(t-r)A}\bar{Z}_r\|_H, \|Z_s\|_H, \|V_{s,r}\|_H\}^q \sum_{n\in\mathbb{N}} \|B(Z_s)e_n\|_H^2\right] && (2.45)\\
&\qquad\le \eta C_0^q \Big\|\max\{1, \|\bar{Z}_r\|_H, \|Z_s\|_H, \|V_{s,r}\|_H\}\Big\|^q_{L^{q+2}(\Omega)} \|B(Z_s)\|^2_{L^{q+2}(\Omega,\mathrm{L}_2(U,H))} && (2.46)\\
&\qquad\le \eta C_0^q \|B\|^2_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}(1+C_{V,T,q+2})^q K^{q+2}_{q+2}. && (2.47)
\end{aligned}
$$

Since the last three estimates do not depend on the integration variable r we immediately get an order of convergence of 1 in time for these terms as well as order $1-\vartheta$ for the first term. More formally the above estimates prove that

$$
\begin{aligned}
&\int_s^t \mathbb{E}\Big[\Big\|\psi_{1,0,0}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})e^{(t-r)A}F(Z_{\lfloor r\rfloor_h})\Big\|_V\Big]\,\mathrm{d}r\\
&\quad+\frac{1}{2}\int_s^t \mathbb{E}\Big[\Big\|\sum_{n\in\mathbb{N}} \psi_{2,0,0}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})(e^{(t-r)A}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})e_n,\\
&\qquad\qquad e^{(t-r)A}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})e_n)\Big\|_V\Big]\,\mathrm{d}r\\
&\quad+\int_s^t \mathbb{E}\Big[\Big\|\sum_{n\in\mathbb{N}} \psi_{1,0,1}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})(e^{(t-r)A}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})e_n, B(Z_s)e_n)\Big\|_V\Big]\,\mathrm{d}r\\
&\quad+\frac{1}{2}\int_s^t \mathbb{E}\Big[\Big\|\sum_{n\in\mathbb{N}} \psi_{0,0,2}(e^{(t-r)A}\bar{Z}_r, Z_s, V_{s,r})(B(Z_s)e_n, B(Z_s)e_n)\Big\|_V\Big]\,\mathrm{d}r\\
&\quad\le \eta C_0^q \frac{C_\vartheta}{1-\vartheta}\sigma\tilde{\sigma}^q_{q+1}K^{q+1}_{q+1}\cdot(t-s)^{1-\vartheta}\\
&\qquad+2\eta C_0^{q+2}\tilde{\sigma}^{q+2}_{q+2}K^{q+2}_{q+2}\cdot(t-s). && (2.48)
\end{aligned}
$$

Now we estimate the first term on the right-hand side of (2.27). In the case that $s>0$, we apply Proposition 1.32 to the mild Itô process $(Y_r)_{r\in[0,s]}$ defined by

$$
Y_r = \begin{pmatrix}\bar{Z}_r\\ Z_r\end{pmatrix},\ \varphi_r = \begin{pmatrix}F(Z_{\lfloor r\rfloor_h})\\ R_rF(Z_{\lfloor r\rfloor_h})\end{pmatrix},\ \Phi_r = \begin{pmatrix}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})\\ R_rG(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h,r})\end{pmatrix},\ S^Y_{t_1,t_2} = \begin{pmatrix}e^{(t_2-t_1)A}\\ S_{t_1,t_2}\end{pmatrix} \tag{2.49}
$$

for $r\in[0,s]$ and $0\le t_1<t_2\le s$, $\check{H}=\hat{H}=H^2$ and the function $f\colon \check{H}\to V$, $(x,y)\mapsto \psi(e^{(t-s)A}x, y, 0)-\psi(x,y,0)$. The function satisfies the polynomial growth assumption, since we have for all $(x,y)\in\check{H}$

$$
\begin{aligned}
\|f(x,y)\|_V &= \|\psi(e^{(t-s)A}x,y,0)-\psi(x,y,0)\|_V\\
&\le \eta C_0^q\max\{1,\|x\|_H,\|y\|_H\}^q\|e^{(t-s)A}x-x\|_H\\
&\le \eta C_0^{q+1}\left(1+\left(\|x\|_H^2+\|y\|_H^2\right)^{\frac{q+1}{2}}\right).
\end{aligned} \tag{2.50}
$$

We obtain

$$\begin{aligned}
&\Big\|\mathbb{E}\Big[\psi\left(e^{(t-s)A}\bar{Z}_s, Z_s, 0\right) - \psi\left(\bar{Z}_s, Z_s, 0\right)\Big]\Big\|_V \\
&\quad\le \Big\|\mathbb{E}\Big[\psi\left(e^{tA}\bar{Z}_0, S_{0,s}Z_0, 0\right) - \psi\left(e^{sA}\bar{Z}_0, S_{0,s}Z_0, 0\right)\Big]\Big\|_V \\
&\qquad + \int_0^s \mathbb{E}\Big[\Big\|\tilde{F}_{r,s,t}(\bar{Z}_r, Z_r, Z_{\lfloor r\rfloor_h})\Big\|_V\Big]\,\mathrm{d}r \\
&\qquad + \int_0^s \mathbb{E}\Big[\Big\|\tilde{B}_{r,s,t}(\bar{Z}_r, Z_r, Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h, r})\Big\|_V\Big]\,\mathrm{d}r
\end{aligned} \tag{2.51}$$

where for $r \in [0,s]$ and $u, v, w, x \in H$ we abbreviated

$$\begin{aligned}
\tilde{F}_{r,s,t}(u,v,w) = {}& \psi_{1,0,0}(e^{(t-r)A}u, S_{r,s}v, 0)e^{(t-r)A}F(w) \\
&- \psi_{1,0,0}(e^{(s-r)A}u, S_{r,s}v, 0)e^{(s-r)A}F(w) \\
&+ \psi_{0,1,0}(e^{(t-r)A}u, S_{r,s}v, 0)S_{r,s}R_rF(w) \\
&- \psi_{0,1,0}(e^{(s-r)A}u, S_{r,s}v, 0)S_{r,s}R_rF(w)
\end{aligned} \tag{2.52}$$

and

$$\begin{aligned}
&\tilde{B}_{r,s,t}(u,v,w,x) \\
&\quad= \frac{1}{2}\sum_{n\in\mathbb{N}} \psi_{2,0,0}(e^{(t-r)A}u, S_{r,s}v, 0)(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n) \\
&\qquad - \frac{1}{2}\sum_{n\in\mathbb{N}} \psi_{2,0,0}(e^{(s-r)A}u, S_{r,s}v, 0)(e^{(s-r)A}G(w,x)e_n, e^{(s-r)A}G(w,x)e_n) \\
&\qquad + \sum_{n\in\mathbb{N}} \psi_{1,1,0}(e^{(t-r)A}u, S_{r,s}v, 0)(e^{(t-r)A}G(w,x)e_n, S_{r,s}R_rG(w,x)e_n) \\
&\qquad - \sum_{n\in\mathbb{N}} \psi_{1,1,0}(e^{(s-r)A}u, S_{r,s}v, 0)(e^{(s-r)A}G(w,x)e_n, S_{r,s}R_rG(w,x)e_n) \\
&\qquad + \frac{1}{2}\sum_{n\in\mathbb{N}} \psi_{0,2,0}(e^{(t-r)A}u, S_{r,s}v, 0)(S_{r,s}R_rG(w,x)e_n, S_{r,s}R_rG(w,x)e_n) \\
&\qquad - \frac{1}{2}\sum_{n\in\mathbb{N}} \psi_{0,2,0}(e^{(s-r)A}u, S_{r,s}v, 0)(S_{r,s}R_rG(w,x)e_n, S_{r,s}R_rG(w,x)e_n).
\end{aligned} \tag{2.53}$$

Now we begin by estimating the first term on the right-hand side of (2.51):

$$\Big\|\mathbb{E}\Big[\psi\left(e^{tA}\bar{Z}_0, S_{0,s}Z_0, 0\right) - \psi\left(e^{sA}\bar{Z}_0, S_{0,s}Z_0, 0\right)\Big]\Big\|_V \tag{2.54}$$

$$\le \eta\mathbb{E}\Big[\max\{1, \|e^{tA}Z_0\|_H, \|e^{sA}Z_0\|_H, \|S_{0,s}Z_0\|_H\}^q\Big\|\left(e^{tA} - e^{sA}\right)Z_0\Big\|_H\Big] \tag{2.55}$$

$$\le \eta C_0^q C_\rho^2 \frac{(t-s)^\rho}{s^\rho}\mathbb{E}\Big[\max\{1, \|Z_0\|_H^{q+1}\}\Big] \le \eta C_0^q C_\rho^2 \frac{(t-s)^\rho}{s^\rho} K_{q+1}^{q+1} \tag{2.56}$$

where we used that $\|e^{tA} - e^{sA}\|_{L(H,H)} \le \|e^{sA}\|_{L(H_{-\rho},H)}\|e^{(t-s)A} - \mathrm{Id}_H\|_{L(H,H_{-\rho})}$ and Assumption 1. For $\tilde{F}$ we get for all $r \in [0,s]$ and $u, v, w \in H$

$$\begin{aligned}
&\Big\|\psi_{1,0,0}\left(e^{(t-r)A}u, S_{r,s}v, 0\right) e^{(t-r)A}F(w) \\
&\qquad - \psi_{1,0,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right) e^{(s-r)A}F(w)\Big\|_V \\
&\quad \leq \Big\|\Big[\psi_{1,0,0}\left(e^{(t-r)A}u, S_{r,s}v, 0\right) - \psi_{1,0,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\Big] e^{(t-r)A}F(w)\Big\|_V \\
&\qquad + \Big\|\psi_{1,0,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right) e^{(s-r)A}\left(e^{(t-s)A} - \mathrm{Id}_H\right) F(w)\Big\|_V
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
&\quad \leq \eta \max\{1, \|e^{(t-r)A}u\|_H \, \|e^{(s-r)A}u\|_H, \|S_{r,s}v\|_H\}^q \\
&\qquad \cdot \Big\|e^{(s-r)A}\left(e^{(t-s)A} - \mathrm{Id}_H\right) u\Big\|_H C_\vartheta (t-r)^{-\vartheta} \|F(w)\|_{H_{-\vartheta}} \\
&\qquad + \eta \max\{1, \|e^{(s-r)A}u\|_H, \|S_{r,s}v\|_H\}^q \\
&\qquad \cdot \Big\|e^{(s-r)A}\left(e^{(t-s)A} - \mathrm{Id}_H\right)\Big\|_{L(H_{-\vartheta},H)} \|F(w)\|_{H_{-\vartheta}}
\end{aligned} \tag{2.58}$$

$$\begin{aligned}
&\quad \leq \eta C_0^q C_\rho^2 C_\vartheta \frac{(t-s)^\rho}{(s-r)^\rho (t-r)^\vartheta} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\{1, \|u\|_H, \|v\|_H\}^{q+1} \max\{1, \|w\|_H\} \\
&\qquad + \eta C_0^q C_\rho C_{\rho+\vartheta} \frac{(t-s)^\rho}{(s-r)^{\rho+\vartheta}} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\{1, \|u\|_H, \|v\|_H\}^q \max\{1, \|w\|_H\}
\end{aligned} \tag{2.59}$$

and

$$\begin{aligned}
&\Big\|\Big[\psi_{0,1,0}\left(e^{(t-r)A}u, S_{r,s}v, 0\right) - \psi_{0,1,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\Big] S_{r,s}R_r F(w)\Big\|_V \\
&\quad \leq \eta C_0^q C_\rho^2 C_\vartheta \frac{(t-s)^\rho}{(s-r)^{\rho+\vartheta}} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\{1, \|u\|_H, \|v\|_H\}^{q+1} \max\{1, \|w\|_H\}\,.
\end{aligned} \tag{2.60}$$

This shows that for all $r \in [0,s]$ and $u, v, w \in H$ it holds that

$$\begin{aligned}
\Big\|\tilde{F}_{r,s,t}(u,v,w)\Big\|_V \leq 3\eta C_0^q C_\rho^2 C_{\rho+\vartheta} C_\vartheta & \frac{(t-s)^\rho}{(s-r)^{\rho+\vartheta}} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \\
& \cdot \max\{1, \|u\|_H, \|v\|_H\}^{q+1} \max\{1, \|w\|_H\}
\end{aligned} \tag{2.61}$$

and because $\rho \in [0, 1-\vartheta)$ we have

$$\begin{aligned}
&\int_0^s \mathbb{E}\Big[\Big\|\tilde{F}_{r,s,t}(\bar{Z}_r, Z_r, Z_{\lfloor r \rfloor_h})\Big\|_V\Big] \, \mathrm{d}r \\
&\qquad \leq 3\eta C_0^q C_\rho^2 C_{\rho+\vartheta} C_\vartheta \sigma K_{q+2}^{q+2} \cdot (t-s)^\rho \frac{s^{1-\vartheta-\rho}}{1-\vartheta-\rho}\,.
\end{aligned} \tag{2.62}$$

Now we estimate $\tilde{B}$. Let $r \in [0,s]$ and $u, v, w, x \in H$. For the first pair of terms we get

$$\begin{aligned}
&\left\| \sum_{n\in\mathbb{N}} \psi_{2,0,0}\left(e^{(t-r)A}u, S_{r,s}v, 0\right)\left(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n\right)\right.\\
&\qquad \left. - \psi_{2,0,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\left(e^{(s-r)A}G(w,x)e_n, e^{(s-r)A}G(w,x)e_n\right)\right\|_V\\
&\quad \le \left\| \sum_{n\in\mathbb{N}} \left[\psi_{2,0,0}\left(e^{(t-r)A}u, S_{r,s}v, 0\right) - \psi_{2,0,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\right]\right.\\
&\qquad \left.\left(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n\right)\right\|_V \qquad (2.63)\\
&\qquad + \left\| \sum_{n\in\mathbb{N}} \psi_{2,0,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\left(\left[e^{(t-r)A} + e^{(s-r)A}\right] G(w,x)e_n,\right.\right.\\
&\qquad \left.\left. e^{(s-r)A}\left(e^{(t-s)A} - \mathrm{Id}_H\right) G(w,x)e_n\right)\right\|_V\\
&\quad \le \eta C_0^{q+2} \max\{1, \|u\|_H, \|v\|_H\}^{q+1} \left\| e^{(s-r)A}\left(e^{(t-s)A} - \mathrm{Id}_H\right)\right\|_{L(H)} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)}\\
&\qquad + 2\eta C_0^{q+1} \max\{1, \|u\|_H, \|v\|_H\}^{q} \left\| e^{(s-r)A}\left(e^{(t-s)A} - \mathrm{Id}_H\right)\right\|_{L(H)} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \qquad (2.64)\\
&\quad \le \eta C_0^{q+2} C_\rho^2 \frac{(t-s)^\rho}{(s-r)^\rho} \max\{1, \|u\|_H, \|v\|_H\}^{q+1} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)}\\
&\qquad + 2\eta C_0^{q+1} C_\rho^2 \frac{(t-s)^\rho}{(s-r)^\rho} \max\{1, \|u\|_H, \|v\|_H\}^{q} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)}\,, \qquad (2.65)
\end{aligned}$$

then for the second pair

$$\begin{aligned}
&\left\| \sum_{n\in\mathbb{N}} \psi_{1,1,0}\left(e^{(t-r)A}u, S_{r,s}v, 0\right)\left(e^{(t-r)A}G(w,x)e_n, S_{r,s}R_rG(w,x)e_n\right)\right.\\
&\qquad \left. - \psi_{1,1,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\left(e^{(s-r)A}G(w,x)e_n, S_{r,s}R_rG(w,x)e_n\right)\right\|_V\\
&\quad \le \left\| \sum_{n\in\mathbb{N}} \left[\psi_{1,1,0}\left(e^{(t-r)A}u, S_{r,s}v, 0\right) - \psi_{1,1,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\right]\right.\\
&\qquad \left.\left(e^{(t-r)A}G(w,x)e_n, S_{r,s}R_rG(w,x)e_n\right)\right\|_V \qquad (2.66)\\
&\qquad + \left\| \sum_{n\in\mathbb{N}} \psi_{1,1,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\right.\\
&\qquad \left.\left(e^{(s-r)A}\left(e^{(t-s)A} - \mathrm{Id}_H\right) G(w,x)e_n, S_{r,s}R_rG(w,x)e_n\right)\right\|_V
\end{aligned}$$

$$\begin{aligned} &\le \eta C_0^{q+2} C_\rho^2 \frac{(t-s)^\rho}{(s-r)^\rho} \max\{1, \|u\|_H, \|v\|_H\}^{q+1} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \\ &\quad + \eta C_0^{q+1} C_\rho^2 \frac{(t-s)^\rho}{(s-r)^\rho} \max\{1, \|u\|_H, \|v\|_H\}^{q} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \end{aligned} \tag{2.67}$$

and for the last pair

$$\begin{aligned} &\Bigg\| \sum_{n\in\mathbb{N}} \Big[\psi_{0,2,0}\left(e^{(t-r)A}u, S_{r,s}v, 0\right) - \psi_{0,2,0}\left(e^{(s-r)A}u, S_{r,s}v, 0\right)\Big] \\ &\qquad (S_{r,s}R_r G(w,x)e_n, S_{r,s}R_r G(w,x)e_n) \Bigg\|_V \\ &\quad \le \eta C_0^{q+2} C_\rho^2 \frac{(t-s)^\rho}{(s-r)^\rho} \max\{1, \|u\|_H, \|v\|_H\}^{q+1} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \,. \end{aligned} \tag{2.68}$$

This shows that for all $r \in [0,s]$ and $u, v, w, x \in H$ it holds that

$$\left\|\tilde{B}_{r,s,t}(u,v,w,x)\right\|_V \le 4\eta C_0^{q+2} C_\rho^2 \frac{(t-s)^\rho}{(s-r)^\rho} \max\{1, \|u\|_H, \|v\|_H\}^{q+1} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \tag{2.69}$$

and thus

$$\begin{aligned} &\int_0^s \mathbb{E}\Big[\Big\|\tilde{B}_{r,s,t}(\bar{Z}_r, Z_r, Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h, r})\Big\|_V\Big] \,\mathrm{d}r \\ &\qquad \le 4\eta C_0^{q+2} C_\rho^2 \tilde{\sigma}_{q+3}^2 K_{q+3}^{q+3} \cdot (t-s)^\rho \frac{s^{1-\rho}}{1-\rho} \,. \end{aligned} \tag{2.70}$$

Inserting (2.56), (2.62) and (2.70) into (2.51) we get that

$$\begin{aligned} &\Big\|\mathbb{E}\Big[\psi\left(e^{(t-s)A}\bar{Z}_s, Z_s, 0\right) - \psi\left(\bar{Z}_s, Z_s, 0\right)\Big]\Big\|_V \\ &\quad \le \eta C_0^q C_\rho^2 K_{q+1}^{q+1} \cdot \frac{(t-s)^\rho}{s^\rho} \\ &\qquad + 3\eta C_0^q C_\rho^2 C_{\rho+\vartheta} C_\vartheta \sigma K_{q+2}^{q+2} \cdot (t-s)^\rho \frac{s^{1-\vartheta-\rho}}{1-\vartheta-\rho} \\ &\qquad + 4\eta C_0^{q+2} C_\rho^2 \tilde{\sigma}_{q+3}^2 K_{q+3}^{q+3} \cdot (t-s)^\rho \frac{s^{1-\rho}}{1-\rho} \\ &\quad \le 4\eta C_0^{q+2} C_\rho^2 C_{\rho+\vartheta} C_\vartheta \sigma \tilde{\sigma}_{q+3}^2 K_{q+3}^{q+3} \cdot (t-s)^\rho \cdot \left[\frac{1}{s^\rho} + \frac{s^{1-\vartheta-\rho}}{1-\vartheta-\rho} + \frac{s^{1-\rho}}{1-\rho}\right] \end{aligned} \tag{2.71}$$

if $s > 0$. To cover the case $s = 0$, we see that in any case we have

$$\begin{aligned} &\Big\|\mathbb{E}\Big[\psi\left(e^{(t-s)A}\bar{Z}_s, Z_s, 0\right) - \psi\left(\bar{Z}_s, Z_s, 0\right)\Big]\Big\|_V \\ &\quad \le \mathbb{E}\Big[\eta \max\Big\{1, \left\|e^{(t-s)A}\bar{Z}_s\right\|_H, \|\bar{Z}_s\|_H, \|Z_s\|_H\Big\}^q \left\|\left(e^{(t-s)A} - \mathrm{Id}_H\right)\bar{Z}_s\right\|_H\Big] \\ &\quad \le \eta C_0^{q+1} K_{q+1}^{q+1} \,. \end{aligned} \tag{2.72}$$

Now we can combine the two estimates using that

$$\min\left\{\frac{1}{(t-s)^\rho}, \frac{1}{s^\rho}\right\} = \max\left\{t-s, s\right\}^{-\rho} \leq \left(\frac{t}{2}\right)^{-\rho} \tag{2.73}$$

to get that

$$\begin{aligned} &\left\| \mathbb{E}\Big[\psi\left(e^{(t-s)A}\bar{Z}_s, Z_s, 0\right) - \psi\left(\bar{Z}_s, Z_s, 0\right)\Big]\right\|_V \\ &\quad \leq 4\eta C_0^{q+2} C_\rho^2 C_{\rho+\vartheta} C_\vartheta \sigma \tilde{\sigma}_{q+3}^2 K_{q+3}^{q+3} \cdot (t-s)^\rho \\ &\qquad \cdot \left[\min\left\{\frac{1}{(t-s)^\rho}, \frac{1}{s^\rho}\right\} + \frac{s^{1-\vartheta-\rho}}{1-\vartheta-\rho} + \frac{s^{1-\rho}}{1-\rho}\right] \\ &\quad \leq 4\eta C_0^{q+2} C_\rho^2 C_{\rho+\vartheta} C_\vartheta \sigma \tilde{\sigma}_{q+3}^2 K_{q+3}^{q+3} \cdot (t-s)^\rho \\ &\qquad \cdot \left[\frac{2^\rho}{t^\rho} + \frac{s^{1-\vartheta-\rho}}{1-\vartheta-\rho} + \frac{s^{1-\rho}}{1-\rho}\right] \end{aligned} \tag{2.74}$$

Combining this with (2.48) and inserting into (2.27) proves that

$$\begin{aligned} &\left\| \mathbb{E}\Big[\psi\left(\bar{Z}_t, Z_s, V_{s,t}\right) - \psi\left(\bar{Z}_s, Z_s, 0\right)\Big]\right\|_V \\ &\quad \leq 4\eta C_0^{q+2} C_\rho^2 C_{\rho+\vartheta} C_\vartheta \sigma \tilde{\sigma}_{q+3}^2 K_{q+3}^{q+3} \cdot (t-s)^\rho \cdot \left[\frac{2^\rho}{t^\rho} + \frac{s^{1-\vartheta-\rho}}{1-\vartheta-\rho} + \frac{s^{1-\rho}}{1-\rho}\right] \\ &\qquad + \eta C_0^q C_\vartheta \sigma \tilde{\sigma}_{q+1}^q K_{q+1}^{q+1} \cdot \frac{(t-s)^{1-\vartheta}}{1-\vartheta} \\ &\qquad + 2\eta C_0^{q+2} \tilde{\sigma}_{q+2}^{q+2} K_{q+2}^{q+2} \cdot (t-s) \\ &\quad \leq 4\eta C_0^{q+2} C_\rho^2 C_{\rho+\vartheta} C_\vartheta \sigma \tilde{\sigma}_{q+3}^{q+2} K_{q+3}^{q+3} \cdot \frac{(t-s)^\rho}{t^\rho} \cdot \left[2^\rho + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho} + \frac{T^{1-\vartheta}}{1-\vartheta} + T\right] \\ &\quad \leq 8\eta C_0^{q+2} C_\rho^2 C_{\rho+\vartheta} C_\vartheta \sigma \tilde{\sigma}_{q+3}^{q+2} K_{q+3}^{q+3} \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right) \cdot \frac{(t-s)^\rho}{t^\rho}. \end{aligned} \tag{2.75}$$

Since $(s,t) \in \Lambda$ were arbitrary, this completes the proof and the constant C can be chosen as

$$\begin{aligned} C &= 8\eta C_0^{q+2} C_\rho^2 C_{\rho+\vartheta} C_\vartheta K_{q+3}^{q+3} \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right) \\ &\quad \cdot \max\{1, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\} \max\{1, \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}, C_G(1 + C_{V,T,q+3})\}^{q+2} \end{aligned} \tag{2.76}$$

which is finite, because under our assumption on q we have $q+3 \leq p$ so that K_{q+3} is finite by Lemma 2.2. □

The second result in this section deals with the convergence of the Milstein approximation process to the auxiliary process as the step size decreases.

Proposition 2.5.
Let $\vartheta \in [0,1)$, $\rho \in [0, 1-\vartheta)$ and $\psi \in C^2(H \times H \times H, V)$ be given. Assume that there exist $q \in [0, p-3]$ and $\eta \in [0,\infty)$ such that for all $x, y_1, y_2, z \in H$ we have

$$\max_{\substack{i,j,k\in\mathbb{N}_0\\ i+j+k\leq 2}} \left\| \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x,y_1,z) - \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x,y_2,z)\right\|_{L^{(i+j+k)}(H,V)} \tag{2.77}$$
$$\leq \eta \max\{1, \|x\|_H, \|y_1\|_H, \|y_2\|_H, \|z\|_H\}^q \|y_1 - y_2\|_H.$$

Then there exists a constant $C \in [0,\infty)$ such that it holds for all $t \in (0,T]$ that

$$\left\|\mathbb{E}\Big[\psi\left(\bar{Z}_t, Z_t, 0\right) - \psi\left(\bar{Z}_t, \bar{Z}_t, 0\right)\Big]\right\|_V \leq C \cdot \frac{h^\rho}{t^\rho}. \tag{2.78}$$

Proof. In the proof we will again use Proposition 1.32 and then estimate all terms using a combination of the assumption (2.77), the estimates on the evolution family (Assumption 1) and standard tools like the triangle and Hölder inequalities. Additionally, we will employ Lemma 2.3 on the distance between the approximation Z and the auxiliary process $\bar{Z}$. For the rest of the proof fix $t \in (0,T]$ and define for $r \in (0,p]$

$$\sigma \coloneqq \max\{1, \|F\|_{\mathrm{Lip}^0(H, H_{-\vartheta})}\}$$

and

$$\tilde{\sigma}_r \coloneqq \max\{1, \|B\|_{\mathrm{Lip}^0(H, \mathrm{L}_2(U,H))}, C_G(1 + C_{V,T,r})\}.$$

We will again use the subscript notation $\psi_{i,j,k}$ to denote the partial derivatives $\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi$.

First, we apply Proposition 1.32 to the mild Itô process $(Y_r)_{r\in[0,t]}$ defined by

$$Y_r = \begin{pmatrix}\bar{Z}_r\\ Z_r\end{pmatrix},\ \varphi_r = \begin{pmatrix}F(Z_{\lfloor r\rfloor_h})\\ R_r F(Z_{\lfloor r\rfloor_h})\end{pmatrix},\ \Phi_r = \begin{pmatrix}G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h, r})\\ R_r G(Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h, r})\end{pmatrix},\ S^Y_{t_1,t_2} = \begin{pmatrix}e^{(t_2-t_1)A}\\ S_{t_1,t_2}\end{pmatrix} \tag{2.79}$$

for $r \in [0,t]$ and $0 \leq t_1 < t_2 \leq t$, $\check{H} = \hat{H} = H^2$ and the function $f\colon \check{H} \to V$, $(x,y) \mapsto \psi(x,y,0) - \psi(x,x,0)$. The function satisfies the polynomial growth assumption, since we have for all $(x,y) \in \check{H}$

$$\begin{aligned}\|f(x,y)\|_V &= \|\psi(x,y,0) - \psi(x,x,0)\|_V\\ &\leq \eta \max\{1, \|x\|_H, \|y\|_H\}^q \|x-y\|_H\\ &\leq 2\eta\left(1 + \left(\|x\|_H^2 + \|y\|_H^2\right)^{\frac{q+1}{2}}\right).\end{aligned} \tag{2.80}$$

This results in

$$\begin{aligned}&\left\|\mathbb{E}\Big[\psi\left(\bar{Z}_t, Z_t, 0\right) - \psi\left(\bar{Z}_t, \bar{Z}_t, 0\right)\Big]\right\|_V\\ &\leq \left\|\mathbb{E}\Big[\psi\left(e^{tA}\bar{Z}_0, S_{0,t}Z_0, 0\right) - \psi\left(e^{tA}\bar{Z}_0, e^{tA}\bar{Z}_0, 0\right)\Big]\right\|_V\\ &\quad + \int_0^t \mathbb{E}\Big[\left\|\tilde{F}_{r,t}(\bar{Z}_r, Z_r, Z_{\lfloor r\rfloor_h})\right\|_V\Big]\,\mathrm{d}r\\ &\quad + \int_0^t \mathbb{E}\Big[\left\|\tilde{B}_{r,t}(\bar{Z}_r, Z_r, Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h, r})\right\|_V\Big]\,\mathrm{d}r\end{aligned} \tag{2.81}$$

where for $r \in [0,t]$ and $u, v, w, x \in H$ we abbreviated

$$\begin{aligned}
&\tilde{F}_{r,t}(u,v,w)\\
&\quad= \Big[\psi_{1,0,0}(e^{(t-r)A}u, S_{r,t}v, 0) - \psi_{1,0,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)\Big]\, e^{(t-r)A}F(w)\\
&\qquad + \psi_{0,1,0}(e^{(t-r)A}u, S_{r,t}v, 0)S_{r,t}R_rF(w)\\
&\qquad - \psi_{0,1,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)e^{(t-r)A}F(w)
\end{aligned} \tag{2.82}$$

and

$$\begin{aligned}
&\tilde{B}_{r,t}(u,v,w,x)\\
&\quad= \frac{1}{2}\sum_{n\in\mathbb{N}} \psi_{2,0,0}(e^{(t-r)A}u, S_{r,t}v, 0)(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n)\\
&\qquad - \frac{1}{2}\sum_{n\in\mathbb{N}} \psi_{2,0,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n)\\
&\qquad + \sum_{n\in\mathbb{N}} \psi_{1,1,0}(e^{(t-r)A}u, S_{r,t}v, 0)(e^{(t-r)A}G(w,x)e_n, S_{r,t}R_rG(w,x)e_n)\\
&\qquad - \sum_{n\in\mathbb{N}} \psi_{1,1,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n)\\
&\qquad + \frac{1}{2}\sum_{n\in\mathbb{N}} \psi_{0,2,0}(e^{(t-r)A}u, S_{r,t}v, 0)(S_{r,t}R_rG(w,x)e_n, S_{r,t}R_rG(w,x)e_n)\\
&\qquad - \frac{1}{2}\sum_{n\in\mathbb{N}} \psi_{0,2,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n)\,.
\end{aligned} \tag{2.83}$$

We begin by estimating the first term on the right-hand side of (2.81):

$$\begin{aligned}
&\Big\| \mathbb{E}\Big[\psi\left(e^{tA}\bar{Z}_0, S_{0,t}Z_0, 0\right) - \psi\left(e^{tA}\bar{Z}_0, e^{tA}\bar{Z}_0, 0\right)\Big]\Big\|_V\\
&\qquad \le \eta\mathbb{E}\Big[\max\{1, \left\|e^{tA}Z_0\right\|_H, \|S_{0,t}Z_0\|_H\}^q \left\|\left(S_{0,t} - e^{tA}\right)Z_0\right\|_H\Big]\\
&\qquad \le \eta C_0^q C_{0,\rho} K_{q+1}^{q+1} \cdot \frac{h^\rho}{t^\rho}
\end{aligned} \tag{2.84}$$

where we used that $\left\|S_{0,t} - e^{tA}\right\|_{L(H,H)} \le C_{0,\rho}h^\rho t^{-\rho}$ by Assumption 1.

For the first integrand, $\tilde{F}$, it holds for all $r \in [0,t]$ and $u, v, w \in H$ that

$$\begin{aligned}
&\left\|\Big[\psi_{1,0,0}(e^{(t-r)A}u, S_{r,t}v, 0) - \psi_{1,0,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)\Big]\, e^{(t-r)A}F(w)\right\|_V\\
&\quad \le \eta\max\{1, \|e^{(t-r)A}u\|_H, \|S_{r,t}v\|_H\}^q \left\|S_{r,t}v - e^{(t-r)A}u\right\|_H\\
&\qquad \cdot \left\|e^{(t-r)A}\right\|_{L(H_{-\vartheta},H)} \|F(w)\|_{H_{-\vartheta}}
\end{aligned} \tag{2.85}$$

$$\begin{aligned}
&\quad \le \eta C_0^q \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\{1, \|u\|_H, \|v\|_H\}^q \max\{1, \|w\|_H\} \left\|e^{(t-r)A}\right\|_{L(H_{-\vartheta},H)}\\
&\qquad \cdot \Big[\left\|S_{r,t} - e^{(t-r)A}\right\|_{L(H,H)} \|v\|_H + \left\|e^{(t-r)A}\right\|_{L(H_{-\rho},H)} \|v - u\|_{H_{-\rho}}\Big]
\end{aligned} \tag{2.86}$$

$$
\begin{aligned}
&\le \eta C_0^q C_\vartheta \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\{1, \|u\|_H, \|v\|_H\}^q \max\{1, \|w\|_H\} \\
&\qquad \cdot \left[\frac{C_{0,\rho} h^\rho}{(t-r)^{\rho+\vartheta}} \|v\|_H + \frac{C_\rho}{(t-r)^{\rho+\vartheta}} \|v-u\|_{H_{-\rho}}\right]
\end{aligned} \tag{2.87}
$$

and

$$
\begin{aligned}
&\Big\|\psi_{0,1,0}(e^{(t-r)A}u, S_{r,t}v, 0) S_{r,t} R_r F(w) \\
&\qquad - \psi_{0,1,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0) e^{(t-r)A} F(w)\Big\|_V \\
&\le \Big\|\psi_{0,1,0}(e^{(t-r)A}u, S_{r,t}v, 0) \left[S_{r,t}R_r - e^{(t-r)A}\right] F(w)\Big\|_V \\
&\qquad + \Big\| \left[\psi_{0,1,0}(e^{(t-r)A}u, S_{r,t}v, 0) - \psi_{0,1,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)\right] \\
&\qquad e^{(t-r)A} F(w)\Big\|_V
\end{aligned} \tag{2.88}
$$

$$
\begin{aligned}
&\le \eta \max\{1, \|e^{(t-r)A}u\|_H, \|S_{r,t}v\|_H\}^q \Big\|S_{r,t}R_r - e^{(t-r)A}\Big\|_{L(H_{-\vartheta},H)} \|F(w)\|_{H_{-\vartheta}} \\
&\qquad + \eta \max\{1, \|e^{(t-r)A}u\|_H, \|S_{r,t}v\|_H\}^q \Big\|S_{r,t}v - e^{(t-r)A}u\Big\|_H \\
&\qquad \cdot \Big\|e^{(t-r)A}\Big\|_{L(H_{-\vartheta},H)} \|F(w)\|_{H_{-\vartheta}}
\end{aligned} \tag{2.89}
$$

$$
\begin{aligned}
&\le \eta C_0^q \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\{1, \|u\|_H, \|v\|_H\}^q \max\{1, \|w\|_H\} C_{\vartheta,0,\rho} \frac{h^\rho}{(t-r)^{\rho+\vartheta}} \\
&\qquad + \eta C_0^q C_\vartheta \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\{1, \|u\|_H, \|v\|_H\}^q \max\{1, \|w\|_H\} \\
&\qquad \cdot \left[\frac{C_{0,\rho} h^\rho}{(t-r)^{\rho+\vartheta}} \|v\|_H + \frac{C_\rho}{(t-r)^{\rho+\vartheta}} \|v-u\|_{H_{-\rho}}\right].
\end{aligned} \tag{2.90}
$$

This shows that for all $r \in [0,t]$ and $u, v, w \in H$ it holds that

$$
\begin{aligned}
\Big\|\tilde{F}_{r,t}(u,v,w)\Big\|_V &\le \eta C_0^q \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\{1, \|u\|_H, \|v\|_H\}^q \max\{1, \|w\|_H\} \\
&\qquad \cdot \left[\frac{C_{\vartheta,0,\rho} h^\rho}{(t-r)^{\rho+\vartheta}} + \frac{2C_\vartheta C_{0,\rho} h^\rho}{(t-r)^{\rho+\vartheta}} \|v\|_H + \frac{2C_\vartheta C_\rho}{(t-r)^{\rho+\vartheta}} \|v-u\|_{H_{-\rho}}\right]
\end{aligned} \tag{2.91}
$$

and thus using Hölder's inequality and Lemma 2.3 we get

$$
\begin{aligned}
&\int_0^t \mathbb{E}\Big[\Big\|\tilde{F}_{r,t}(\bar{Z}_r, Z_r, Z_{\lfloor r \rfloor_h})\Big\|_V\Big] \,\mathrm{d}r \\
&\qquad \le \eta C_0^q \sigma \int_0^t \frac{C_{\vartheta,0,\rho} h^\rho}{(t-r)^{\rho+\vartheta}} K_{q+1}^{q+1} + \frac{2C_\vartheta C_{0,\rho} h^\rho}{(t-r)^{\rho+\vartheta}} K_{q+2}^{q+2} \\
&\qquad\qquad + \frac{2C_\vartheta C_\rho}{(t-r)^{\rho+\vartheta}} K_{q+2}^{q+1} \|Z_r - \bar{Z}_r\|_{L^{q+2}(\Omega, H_{-\rho})} \,\mathrm{d}r \\
&\qquad \le 2\eta C_0^q \sigma \left[C_{\vartheta,0,\rho} + C_\vartheta C_{0,\rho} + C_\vartheta C_\rho A_{T,\rho,q+2}\right] K_{q+2}^{q+2} \cdot h^\rho \frac{t^{1-\vartheta-\rho}}{1-\vartheta-\rho}.
\end{aligned} \tag{2.92}
$$

Next we estimate the second integrand, $\tilde{B}$, by observing that for all $r \in [0,t]$ and $u, v, w, x \in H$ it holds that

$$
\begin{aligned}
&\Bigg\| \sum_{n\in\mathbb{N}} \Big[\psi_{2,0,0}(e^{(t-r)A}u, S_{r,t}v, 0) - \psi_{2,0,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)\Big] \\
&\qquad \Big(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n\Big) \Bigg\|_V \qquad (2.93)\\
&\le \eta C_0^q \max\{1, \|u\|_H, \|v\|_H\}^q \Big\|S_{r,t}v - e^{(t-r)A}u\Big\|_H \Big\|e^{(t-r)A}G(w,x)\Big\|^2_{\mathrm{L}_2(U,H)} \\
&\le \eta C_0^{q+2} \max\{1, \|u\|_H, \|v\|_H\}^q \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \\
&\qquad \left[\frac{C_{0,\rho}h^\rho}{(t-r)^\rho}\|v\|_H + \frac{C_\rho}{(t-r)^\rho}\|v-u\|_{H_{-\rho}}\right], \qquad (2.94)
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg\| \sum_{n\in\mathbb{N}} \psi_{1,1,0}(e^{(t-r)A}u, S_{r,t}v, 0)\Big(e^{(t-r)A}G(w,x)e_n, S_{r,t}R_rG(w,x)e_n\Big) \\
&\qquad - \psi_{1,1,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)\Big(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n\Big) \Bigg\|_V \\
&\le \Bigg\| \sum_{n\in\mathbb{N}} \psi_{1,1,0}(e^{(t-r)A}u, S_{r,t}v, 0)\Big(e^{(t-r)A}G(w,x)e_n, \\
&\qquad \Big[S_{r,t}R_r - e^{(t-r)A}\Big]G(w,x)e_n\Big)\Bigg\|_V \qquad (2.95)\\
&\qquad + \Bigg\| \sum_{n\in\mathbb{N}} \Big[\psi_{1,1,0}(e^{(t-r)A}u, S_{r,t}v, 0) - \psi_{1,1,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)\Big] \\
&\qquad \Big(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n\Big) \Bigg\|_V \\
&\le \eta C_0^{q+1} \max\{1, \|u\|_H, \|v\|_H\}^q \Big\|S_{r,t}R_r - e^{(t-r)A}\Big\|_{L(H,H)} \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \\
&\qquad + \eta C_0^{q+2} \max\{1, \|u\|_H, \|v\|_H\}^q \Big\|S_{r,t}v - e^{(t-r)A}u\Big\|_H \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \qquad (2.96)\\
&\le \eta C_0^{q+1} C_{0,0,\rho}\frac{h^\rho}{(t-r)^\rho} \max\{1, \|u\|_H, \|v\|_H\}^q \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \\
&\qquad + \eta C_0^{q+2} \max\{1, \|u\|_H, \|v\|_H\}^q \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \qquad (2.97)\\
&\qquad \cdot \left[\frac{C_{0,\rho}h^\rho}{(t-r)^\rho}\|v\|_H + \frac{C_\rho}{(t-r)^\rho}\|v-u\|_{H_{-\rho}}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\| \sum_{n\in\mathbb{N}} \psi_{0,2,0}(e^{(t-r)A}u, S_{r,t}v, 0)\,(S_{r,t}R_rG(w,x)e_n, S_{r,t}R_rG(w,x)e_n)\\
&\qquad\qquad - \psi_{0,2,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)\Big(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n\Big)\Big\|_V\\
&\quad\le \Big\| \sum_{n\in\mathbb{N}} \psi_{0,2,0}(e^{(t-r)A}u, S_{r,t}v, 0)\Big(\Big[S_{r,t}R_r + e^{(t-r)A}\Big]G(w,x)e_n,\\
&\qquad \Big[S_{r,t}R_r - e^{(t-r)A}\Big]G(w,x)e_n\Big)\Big\|_V \qquad (2.98)\\
&\qquad + \Big\| \sum_{n\in\mathbb{N}} \Big[\psi_{0,2,0}(e^{(t-r)A}u, S_{r,t}v, 0) - \psi_{0,2,0}(e^{(t-r)A}u, e^{(t-r)A}u, 0)\Big]\\
&\qquad \Big(e^{(t-r)A}G(w,x)e_n, e^{(t-r)A}G(w,x)e_n\Big)\Big\|_V\\
&\quad\le \eta C_0^q \max\{1, \|u\|_H, \|v\|_H\}^q \|G(w,x)\|^2_{\mathrm{L}_2(U,H)}\\
&\qquad \Big\|S_{r,t}R_r + e^{(t-r)A}\Big\|_{L(H,H)} \Big\|S_{r,t}R_r - e^{(t-r)A}\Big\|_{L(H,H)} \qquad (2.99)\\
&\qquad + \eta C_0^{q+2} \max\{1, \|u\|_H, \|v\|_H\}^q \Big\|S_{r,t}v - e^{(t-r)A}u\Big\|_H \|G(w,x)\|^2_{\mathrm{L}_2(U,H)}\\
&\quad\le 2\eta C_0^{q+1} C_{0,0,\rho}\frac{h^\rho}{(t-r)^\rho} \max\{1, \|u\|_H, \|v\|_H\}^q \|G(w,x)\|^2_{\mathrm{L}_2(U,H)}\\
&\qquad + \eta C_0^{q+2} \max\{1, \|u\|_H, \|v\|_H\}^q \|G(w,x)\|^2_{\mathrm{L}_2(U,H)} \qquad (2.100)\\
&\qquad \cdot \left[\frac{C_{0,\rho}h^\rho}{(t-r)^\rho}\|v\|_H + \frac{C_\rho}{(t-r)^\rho}\|v-u\|_{H_{-\rho}}\right].
\end{aligned}
$$

This shows that for all $r \in [0,t]$ and $u, v, w, x \in H$ we have

$$
\begin{aligned}
\Big\|\tilde{B}_{r,t}(u,v,w,x)\Big\|_V &\le 2\eta C_0^{q+2} \max\{1, \|u\|_H, \|v\|_H\}^q \|G(w,x)\|^2_{\mathrm{L}_2(U,H)}\\
&\quad \cdot \left[\frac{C_{0,0,\rho}h^\rho}{(t-r)^\rho} + \frac{C_{0,\rho}h^\rho}{(t-r)^\rho}\|v\|_H + \frac{C_\rho}{(t-r)^\rho}\|v-u\|_{H_{-\rho}}\right]
\end{aligned} \tag{2.101}
$$

and thus using Hölder's inequality and Lemma 2.3 we get

$$
\begin{aligned}
&\int_0^t \mathbb{E}\Big[\Big\|\tilde{B}_{r,t}(\bar{Z}_r, Z_r, Z_{\lfloor r\rfloor_h}, V_{\lfloor r\rfloor_h, r})\Big\|_V\Big]\,\mathrm{d}r\\
&\quad\le 2\eta C_0^{q+2} \int_0^t \frac{C_{0,0,\rho}h^\rho}{(t-r)^\rho}\tilde{\sigma}^2_{q+2}K^{q+2}_{q+2} + \frac{C_{0,\rho}h^\rho}{(t-r)^\rho}\tilde{\sigma}^2_{q+3}K^{q+3}_{q+3}\\
&\qquad + \frac{C_\rho}{(t-r)^\rho}\tilde{\sigma}^2_{q+3}K^{q+2}_{q+3}\|v-u\|_{L^{q+3}(\Omega,H_{-\rho})}\,\mathrm{d}r \qquad (2.102)\\
&\quad\le 2\eta C_0^{q+2}\tilde{\sigma}^2_{q+3}\left[C_{0,0,\rho} + C_{0,\rho} + C_\rho A_{T,\rho,q+3}\right]K^{q+3}_{q+3} \cdot h^\rho \frac{t^{1-\rho}}{1-\rho}.
\end{aligned}
$$

Now we can insert the estimates (2.84), (2.92) and (2.102) into (2.81) to get that

$$
\begin{aligned}
&\left\|\mathbb{E}\left[\psi\left(\bar{Z}_t, Z_t, 0\right)-\psi\left(\bar{Z}_t, \bar{Z}_t, 0\right)\right]\right\|_V \\
&\quad\leq \eta C_0^q C_{0,\rho} K_{q+1}^{q+1}\cdot\frac{h^\rho}{t^\rho} \\
&\qquad + 2\eta C_0^q \sigma\left[C_{\vartheta,0,\rho}+C_\vartheta C_{0,\rho}+C_\vartheta C_\rho A_{T,\rho,q+2}\right] K_{q+2}^{q+2}\cdot h^\rho \frac{t^{1-\vartheta-\rho}}{1-\vartheta-\rho} && (2.103)\\
&\qquad + 2\eta C_0^{q+2}\tilde{\sigma}_{q+3}^2\left[C_{0,0,\rho}+C_{0,\rho}+C_\rho A_{T,\rho,q+3}\right] K_{q+3}^{q+3}\cdot h^\rho\frac{t^{1-\rho}}{1-\rho} \\
&\quad\leq 2\eta C_0^{q+2}\sigma\tilde{\sigma}_{q+3}^2\left[C_{0,0,\rho}+C_{\vartheta,0,\rho}+C_\vartheta C_{0,\rho}+C_\vartheta C_\rho A_{T,\rho,q+3}\right] K_{q+3}^{q+3} \\
&\qquad\cdot h^\rho\left(\frac{1}{t^\rho}+\frac{t^{1-\vartheta-\rho}}{1-\vartheta-\rho}+\frac{t^{1-\rho}}{1-\rho}\right) && (2.104)\\
&\quad\leq 2\eta C_0^{q+2}\sigma\tilde{\sigma}_{q+3}^2\left[C_{0,0,\rho}+C_{\vartheta,0,\rho}+C_\vartheta C_{0,\rho}+C_\vartheta C_\rho A_{T,\rho,q+3}\right] K_{q+3}^{q+3} \\
&\qquad\cdot\left(1+\frac{T^{1-\vartheta}}{1-\vartheta-\rho}+\frac{T}{1-\rho}\right)\cdot\frac{h^\rho}{t^\rho}\,. && (2.105)
\end{aligned}
$$

Since $t \in (0, T]$ was arbitrary, this completes the proof and the constant C can be chosen as

$$
\begin{aligned}
C = {}& 2\eta C_0^{q+2}\left[C_{0,0,\rho}+C_{\vartheta,0,\rho}+C_\vartheta C_{0,\rho}+C_\vartheta C_\rho A_{T,\rho,q+3}\right] K_{q+3}^{q+3}\left(1+\frac{T^{1-\vartheta}}{1-\vartheta-\rho}+\frac{T}{1-\rho}\right) \\
&\cdot\max\{1,\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\}\max\{1,\|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}, C_G(1+C_{V,T,q+3})\}^2
\end{aligned}
\tag{2.106}
$$

which is finite, because under our assumption on q we have $q+3 \leq p$ such that K_{q+3} is finite by Lemma 2.2. □

2.4 Bounds Related to the Kolmogorov Backward Equation

Let u be the solution to the Kolmogorov backward equation associated to X (see (1.13)). By Assumption 3 and Theorem 1.37 this solution exists and has the stated properties. We will now see that the functions $\psi(x,y,z) = (\frac{\partial}{\partial x}u)(t,x)F(y)$ and $\psi(x,y,z) = \sum_{n\in\mathbb{N}}(\frac{\partial^2}{\partial x^2}u)(t,x)\left(G(y,z)e_n, G(y,z)e_n\right)$, which are the two terms on the right-hand side of the Kolmogorov backward equation, satisfy the local Lipschitz estimates needed for Propositions 2.4 and 2.5. Since we need bounds on all second derivatives, it follows that u has to have at least four derivatives.

The proofs rely on the fundamental theorem of calculus (whereby “Lipschitz equals bounded derivative”) and make use of the constants in Theorem 1.37. In order for these constants to be finite, we would have to make the stronger assumption required by Theorem 1.37 that our coefficients take values in H_1 instead of Assumption 3. This will be done in the next section, where we have to assume that the coefficients take values even in H_2.

Lemma 2.6.
Let $\vartheta \in [0, \frac{1}{2})$ *and* $t \in [0, T)$ *and define the functions*

$$\psi\colon H \times H \times H \to V, \quad (x, y, z) \mapsto \Big(\frac{\partial}{\partial x} u\Big)(t, x) F(y) \tag{2.107}$$

and $\phi\colon H \times H \to V$, $(x, z) \mapsto \psi(x, x, z)$. *Then we have that* $\psi \in C^3(H \times H \times H, V)$ *and* $\phi \in C^3(H \times H, V)$ *and for all* $x, x_1, x_2, y, y_1, y_2, z \in H$ *it holds that*

$$\begin{aligned} &\max_{\substack{i,j,k \in \mathbb{N}_0 \\ i+j+k \leq 2}} \left\| \left(\frac{\partial^{i+j+k}}{\partial x^i \, \partial y^j \, \partial z^k} \psi \right) (x_1, y, z) - \left(\frac{\partial^{i+j+k}}{\partial x^i \, \partial y^j \, \partial z^k} \psi \right) (x_2, y, z) \right\|_{L^{(i+j+k)}(H,V)} \\ &\quad \leq \|F\|_{C^2_b(H, H_{-\vartheta})} (c_{\vartheta,0} + c_{\vartheta,0,0} + c_{\vartheta,0,0,0}) \max\{1, \|y\|_H\} \frac{\|x_1 - x_2\|_H}{(T-t)^{\vartheta}}, \end{aligned} \tag{2.108}$$

$$\begin{aligned} &\max_{\substack{i,j,k \in \mathbb{N}_0 \\ i+j+k \leq 2}} \left\| \left(\frac{\partial^{i+j+k}}{\partial x^i \, \partial y^j \, \partial z^k} \psi \right) (x, y_1, z) - \left(\frac{\partial^{i+j+k}}{\partial x^i \, \partial y^j \, \partial z^k} \psi \right) (x, y_2, z) \right\|_{L^{(i+j+k)}(H,V)} \\ &\quad \leq \|F\|_{C^3_b(H, H_{-\vartheta})} (c_{\vartheta} + c_{\vartheta,0} + c_{\vartheta,0,0}) \frac{\|y_1 - y_2\|_H}{(T-t)^{\vartheta}} \end{aligned} \tag{2.109}$$

as well as

$$\begin{aligned} &\max_{\substack{i,j \in \mathbb{N}_0 \\ i+j \leq 2}} \left\| \left(\frac{\partial^{i+j}}{\partial x^i \, \partial z^j} \phi \right) (x_1, z) - \left(\frac{\partial^{i+j}}{\partial x^i \, \partial z^j} \phi \right) (x_2, z) \right\|_{L^{(i+j)}(H,V)} \\ &\quad \leq 3 \|F\|_{C^3_b(H, H_{-\vartheta})} (c_{\vartheta} + c_{\vartheta,0} + c_{\vartheta,0,0} + c_{\vartheta,0,0,0}) \max\{1, \|x_1\|_H, \|x_2\|_H\} \frac{\|x_1 - x_2\|_H}{(T-t)^{\vartheta}}. \end{aligned} \tag{2.110}$$

Proof. By Theorem 1.37 we have that $[x \mapsto u(t, x)] \in C^4_b(H, V)$ and this ensures the claimed differentiability of ψ and ϕ. Since ψ and ϕ do not actually depend on the third parameter, all the derivatives with respect to z vanish. The rest of the lemma is thus equal to [JK21, Lemma 6.1]. ☐

Lemma 2.7.
Let $t \in [0, T)$ *and define the functions*

$$\psi\colon H \times H \times H \to V, \quad (x, y, z) \mapsto \sum_{n \in \mathbb{N}} \Big(\frac{\partial^2}{\partial x^2} u\Big)(t, x) (G(y, z) e_n, G(y, z) e_n)$$

and $\phi\colon H \times H \to V$, $(x, z) \mapsto \psi(x, x, z)$. *Then we have that* $\psi \in C^2(H \times H \times H, V)$ *and* $\phi \in C^2(H \times H, V)$ *and for all* $x, x_1, x_2, y, y_1, y_2, z \in H$ *it holds that*

$$\max_{\substack{i,j,k\in\mathbb{N}_0\\ i+j+k\leq 2}} \left\| \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x_1,y,z) - \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x_2,y,z)\right\|_{L^{(i+j+k)}(H,V)} \\ \leq 4\,(c_{0,0,0}+c_{0,0,0,0}+\tilde{c}_{0,0,0,0})\,C_G^2 \max\{1,\|y\|_H,\|z\|_H\}^2\|x_1-x_2\|_H, \tag{2.111}$$

$$\max_{\substack{i,j,k\in\mathbb{N}_0\\ i+j+k\leq 2}} \left\| \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x,y_1,z) - \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x,y_2,z)\right\|_{L^{(i+j+k)}(H,V)} \\ \leq 8\,(c_{0,0}+c_{0,0,0}+c_{0,0,0,0})\,C_G^2 \max\{1,\|y_1\|_H,\|y_2\|_H,\|z\|_H\}^2\|y_1-y_2\|_H \tag{2.112}$$

as well as

$$\max_{\substack{i,j\in\mathbb{N}_0\\ i+j\leq 2}} \left\| \left(\frac{\partial^{i+j}}{\partial x^i\,\partial z^j}\phi\right)(x_1,z) - \left(\frac{\partial^{i+j}}{\partial x^i\,\partial z^j}\phi\right)(x_2,z)\right\|_{L^{(i+j)}(H,V)} \\ \leq 14\,(c_{0,0}+c_{0,0,0}+c_{0,0,0,0}+\tilde{c}_{0,0,0,0})\,C_G^2 \max\{1,\|x_1\|_H,\|x_2\|_H,\|z\|_H\}^2\|x_1-x_2\|_H. \tag{2.113}$$

Proof. By Theorem 1.37 we have again that $[x\mapsto u(t,x)]\in C_b^4(H,V)$ and this ensures the claimed differentiability of ψ and ϕ.

We will now use the fundamental theorem of calculus in Banach spaces to reduce the proof to showing estimates for one further derivative. More formally, we use that for $0\leq i\leq 1$ and $0\leq j,k\leq 2$ with $i+j+k\leq 2$ we have for all $x_1,x_2,y,z\in H$

$$\left\| \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x_1,y,z) - \left(\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\psi\right)(x_2,y,z)\right\|_{L^{(i+j+k)}(H,V)} \\ \leq \int_0^1 \left\| \left(\frac{\partial^{i+j+k+1}}{\partial x^{i+1}\,\partial y^j\,\partial z^k}\psi\right)(x_2+r(x_1-x_2),y,z)\right\|_{L^{(i+1+j+k)}(H,V)} \mathrm{d}r \|x_1-x_2\|_H \tag{2.114}$$

and we now provide estimates for

$$\left\| \left(\frac{\partial^{i+j+k+1}}{\partial x^{i+1}\,\partial y^j\,\partial z^k}\psi\right)(x_2+r(x_1-x_2),y,z)(v_1,\ldots,v_{i+1+j+k})\right\|_V \tag{2.115}$$

where $v_l\in H$ with $\|v_l\|_H\leq 1$ for $1\leq l\leq i+1+j+k$. The "$i=2$" case, in contrast, will be handled directly using the Lipschitz property (1.15). Together, these will then imply (2.111). The estimates (2.112) and (2.113) will be handled analogously.

In the following, we will use the subscript notation $u_{0,i}$ to denote the partial derivatives $\frac{\partial^i}{\partial x^i}u$.

Let now $x,y,z\in H$ and $v_1,v_2,v_3\in H$ with $\|v_1\|_H,\|v_2\|_H,\|v_3\|_H\leq 1$. Then using Theorem 1.37, Assumption 2 and the fact that $\sum_{n\in\mathbb{N}}\|Te_n\|_H\|Se_n\|_H\leq\|T\|_{\mathrm{L}_2(U,H)}\|S\|_{\mathrm{L}_2(U,H)}$ for $S,T\in\mathrm{L}_2(U,H)$ it holds that

$$\begin{aligned}\left\|\left(\frac{\partial}{\partial x}\psi\right)(x,y,z)v_1\right\|_V &= \left\|\sum_{n\in\mathbb{N}} u_{0,3}(t,x)(G(y,z)e_n, G(y,z)e_n, v_1)\right\|_V \\ &\leq c_{0,0,0}\sum_{n\in\mathbb{N}}\|G(y,z)e_n\|_H^2\|v_1\|_H \\ &\leq c_{0,0,0}\|G(y,z)\|^2_{\mathrm{L}_2(U,H)} \\ &\leq c_{0,0,0}C_G^2\max\{1,\|y\|_H,\|z\|_H\}^2, \end{aligned} \tag{2.116}$$

$$\begin{aligned}\left\|\left(\frac{\partial^2}{\partial x^2}\psi\right)(x,y,z)(v_1,v_2)\right\|_V &= \left\|\sum_{n\in\mathbb{N}} u_{0,4}(t,x)(G(y,z)e_n, G(y,z)e_n, v_1, v_2)\right\|_V \\ &\leq c_{0,0,0,0}C_G^2\max\{1,\|y\|_H,\|z\|_H\}^2, \end{aligned} \tag{2.117}$$

$$\begin{aligned}\left\|\left(\frac{\partial^2}{\partial x\,\partial z}\psi\right)(x,y,z)(v_1,v_2)\right\|_V &= \left\|2\sum_{n\in\mathbb{N}} u_{0,3}(t,x)(G(y,z)e_n, \left(\frac{\partial}{\partial z}G\right)(y,z)v_1e_n, v_2)\right\|_V \\ &\leq 2c_{0,0,0}C_G^2\max\{1,\|y\|_H,\|z\|_H\}^2, \end{aligned} \tag{2.118}$$

$$\begin{aligned}&\left\|\left(\frac{\partial^3}{\partial x\,\partial z^2}\psi\right)(x,y,z)(v_1,v_2,v_3)\right\|_V \\ &= \Bigg\|2\sum_{n\in\mathbb{N}} u_{0,3}(t,x)(\left(\frac{\partial}{\partial z}G\right)(y,z)v_2b, \left(\frac{\partial}{\partial z}G\right)(y,z)v_1e_n, v_3) \\ &\qquad + u_{0,3}(t,x)(G(y,z)e_n, \left(\frac{\partial^2}{\partial z^2}G\right)(y,z)(v_1,v_2)e_n, v_3)\Bigg\|_V \\ &\leq 4c_{0,0,0}C_G^2\max\{1,\|y\|_H,\|z\|_H\}^2, \end{aligned} \tag{2.119}$$

$$\begin{aligned}&\left\|\left(\frac{\partial^3}{\partial x^2\,\partial z}\psi\right)(x,y,z)(v_1,v_2,v_3)\right\|_V \\ &= \left\|2\sum_{n\in\mathbb{N}} u_{0,4}(t,x)(G(y,z)e_n, \left(\frac{\partial}{\partial z}G\right)(y,z)v_1e_n, v_2, v_3)\right\|_V \\ &\leq 2c_{0,0,0,0}C_G^2\max\{1,\|y\|_H,\|z\|_H\}^2, \end{aligned} \tag{2.120}$$

$$\begin{aligned}\left\|\left(\frac{\partial^2}{\partial x\,\partial y}\psi\right)(x,y,z)(v_1,v_2)\right\|_V &= \left\|2\sum_{n\in\mathbb{N}} u_{0,3}(t,x)(G(y,z)e_n, \left(\frac{\partial}{\partial y}G\right)(y,z)v_1e_n, v_2)\right\|_V \\ &\leq 2c_{0,0,0}C_G^2\max\{1,\|y\|_H,\|z\|_H\}^2, \end{aligned} \tag{2.121}$$

$$\begin{aligned}
&\left\| \left(\frac{\partial^3}{\partial x\, \partial y^2} \psi \right) (x, y, z)(v_1, v_2, v_3) \right\|_V \\
&\quad = \Bigg\| 2 \sum_{n \in \mathbb{N}} u_{0,3}(t, x)(\left(\frac{\partial}{\partial y} G \right) (y, z) v_2 b, \left(\frac{\partial}{\partial y} G \right) (y, z) v_1 e_n, v_3) \\
&\qquad\quad + u_{0,3}(t, x)(G(y, z) e_n, \left(\frac{\partial^2}{\partial y^2} G \right) (y, z)(v_1, v_2) e_n, v_3) \Bigg\|_V \\
&\quad \leq 4 c_{0,0,0} C_G^2 \max\{1, \|y\|_H, \|z\|_H\}^2,
\end{aligned} \tag{2.122}$$

$$\begin{aligned}
&\left\| \left(\frac{\partial^3}{\partial x^2\, \partial y} \psi \right) (x, y, z)(v_1, v_2, v_3) \right\|_V \\
&\quad = \left\| 2 \sum_{n \in \mathbb{N}} u_{0,4}(t, x)(G(y, z) e_n, \left(\frac{\partial}{\partial y} G \right) (y, z) v_1 e_n, v_2, v_3) \right\|_V \\
&\quad \leq 2 c_{0,0,0,0} C_G^2 \max\{1, \|y\|_H, \|z\|_H\}^2,
\end{aligned} \tag{2.123}$$

$$\begin{aligned}
&\left\| \left(\frac{\partial^3}{\partial x\, \partial y\, \partial z} \psi \right) (x, y, z)(v_1, v_2, v_3) \right\|_V \\
&\quad = \Bigg\| 2 \sum_{n \in \mathbb{N}} u_{0,3}(t, x)(\left(\frac{\partial}{\partial y} G \right) (y, z) v_2 b, \left(\frac{\partial}{\partial z} G \right) (y, z) v_1 e_n, v_3) \\
&\qquad\quad + u_{0,3}(t, x)(G(y, z) e_n, \left(\frac{\partial^2}{\partial y\, \partial z} G \right) (y, z)(v_1, v_2) e_n, v_3) \Bigg\|_V \\
&\quad \leq 4 c_{0,0,0} C_G^2 \max\{1, \|y\|_H, \|z\|_H\}^2.
\end{aligned} \tag{2.124}$$

Now we use the Lipschitz property (1.15) to get that for $x_1, x_2, y, z \in H$ and $v_1, v_2 \in H$ with $\|v_1\|_H, \|v_2\|_H \leq 1$ it holds that

$$\begin{aligned}
&\left\| \left(\frac{\partial^2}{\partial x^2} \psi \right) (x_1, y, z)(v_1, v_2) - \left(\frac{\partial^2}{\partial x^2} \psi \right) (x_2, y, z)(v_1, v_2) \right\|_V \\
&\quad = \left\| \sum_{n \in \mathbb{N}} \left[u_{0,4}(t, x_1) - u_{0,4}(t, x_2) \right] (G(y, z) e_n, G(y, z) e_n, v_1, v_2) \right\|_V \\
&\quad \leq \tilde{c}_{0,0,0,0} C_G^2 \max\{1, \|y\|_H, \|z\|_H\}^2 \|x_1 - x_2\|_H.
\end{aligned} \tag{2.125}$$

The fundamental theorem of calculus and (2.116)–(2.124) as well as (2.125) prove the first assertion (2.111).

To prove the second assertion, consider again $x, y, z \in H$ and $v_1, v_2, v_3 \in H$ with $\|v_1\|_H, \|v_2\|_H, \|v_3\|_H \leq 1$. Then we estimate as before

$$\begin{aligned}\left\|\left(\frac{\partial}{\partial y}\psi\right)(x,y,z)v_1\right\|_V &= \left\|2\sum_{n\in\mathbb{N}} u_{0,2}(t,x)(G(y,z)e_n, \left(\frac{\partial}{\partial y}G\right)(y,z)v_1e_n)\right\|_V \\ &\leq 2c_{0,0}C_G^2 \max\{1, \|y\|_H, \|z\|_H\}^2,\end{aligned}\tag{2.126}$$

$$\begin{aligned}\left\|\left(\frac{\partial^2}{\partial y^2}\psi\right)(x,y,z)(v_1,v_2)\right\|_V &= \Bigg\|2\sum_{n\in\mathbb{N}} u_{0,2}(t,x)(\left(\frac{\partial}{\partial y}G\right)(y,z)v_2b, \left(\frac{\partial}{\partial y}G\right)(y,z)v_1e_n) \\ &\quad + u_{0,2}(t,x)(G(y,z)e_n, \left(\frac{\partial^2}{\partial y^2}G\right)(y,z)(v_1,v_2)e_n)\Bigg\|_V \\ &\leq 4c_{0,0}C_G^2 \max\{1, \|y\|_H, \|z\|_H\}^2,\end{aligned}\tag{2.127}$$

$$\begin{aligned}\left\|\left(\frac{\partial^2}{\partial y\,\partial z}\psi\right)(x,y,z)(v_1,v_2)\right\|_V &= \Bigg\|2\sum_{n\in\mathbb{N}} u_{0,2}(t,x)(\left(\frac{\partial}{\partial y}G\right)(y,z)v_2b, \left(\frac{\partial}{\partial z}G\right)(y,z)v_1e_n) \\ &\quad + u_{0,2}(t,x)(G(y,z)e_n, \left(\frac{\partial^2}{\partial y\,\partial z}G\right)(y,z)(v_1,v_2)e_n)\Bigg\|_V \\ &\leq 4c_{0,0}C_G^2 \max\{1, \|y\|_H, \|z\|_H\}^2,\end{aligned}\tag{2.128}$$

$$\begin{aligned}&\left\|\left(\frac{\partial^3}{\partial y^3}\psi\right)(x,y,z)(v_1,v_2,v_3)\right\|_V \\ &= \Bigg\|2\sum_{n\in\mathbb{N}} u_{0,2}(t,x)(\left(\frac{\partial}{\partial y}G\right)(y,z)v_1e_n, \left(\frac{\partial^2}{\partial y^2}G\right)(y,z)(v_2,v_3)e_n) \\ &\quad + u_{0,2}(t,x)(\left(\frac{\partial}{\partial y}G\right)(y,z)v_2b, \left(\frac{\partial^2}{\partial y^2}G\right)(y,z)(v_1,v_3)e_n) \\ &\quad + u_{0,2}(t,x)(\left(\frac{\partial}{\partial y}G\right)(y,z)v_3b, \left(\frac{\partial^2}{\partial y^2}G\right)(y,z)(v_1,v_2)e_n) \\ &\quad + u_{0,2}(t,x)(G(y,z)e_n, \left(\frac{\partial^3}{\partial y^3}G\right)(y,z)(v_1,v_2,v_3)e_n)\Bigg\|_V \\ &\leq 8c_{0,0}C_G^2 \max\{1, \|y\|_H, \|z\|_H\}^2,\end{aligned}\tag{2.129}$$

$$\begin{aligned}&\left\|\left(\frac{\partial^3}{\partial y^2\,\partial z}\psi\right)(x,y,z)(v_1,v_2,v_3)\right\|_V \\ &= \Bigg\|2\sum_{n\in\mathbb{N}} u_{0,2}(t,x)(\left(\frac{\partial}{\partial z}G\right)(y,z)v_1e_n, \left(\frac{\partial^2}{\partial y^2}G\right)(y,z)(v_2,v_3)e_n) \\ &\quad + u_{0,2}(t,x)(\left(\frac{\partial}{\partial y}G\right)(y,z)v_2b, \left(\frac{\partial^2}{\partial y\,\partial z}G\right)(y,z)(v_1,v_3)e_n)\end{aligned}$$

$$\begin{aligned}
&+ u_{0,2}(t,x)(\left(\frac{\partial}{\partial y}G\right)(y,z)v_3 b, \left(\frac{\partial^2}{\partial y\,\partial z}G\right)(y,z)(v_1,v_2)e_n) \\
&+ u_{0,2}(t,x)(G(y,z)e_n, \left(\frac{\partial^3}{\partial y^2\,\partial z}G\right)(y,z)(v_1,v_2v_3)e_n)\Bigg\|_V \\
&\leq 8c_{0,0}C_G^2 \max\{1,\|y\|_H,\|z\|_H\}^2, \qquad (2.130)
\end{aligned}$$

$$\begin{aligned}
&\left\|\left(\frac{\partial^3}{\partial y\,\partial z^2}\psi\right)(x,y,z)(v_1,v_2,v_3)\right\|_V \\
&= \Bigg\| 2\sum_{n\in\mathbb{N}} u_{0,2}(t,x)(\left(\frac{\partial}{\partial z}G\right)(y,z)v_1e_n, \left(\frac{\partial^2}{\partial y\,\partial z}G\right)(y,z)(v_2,v_3)e_n) \\
&\quad + u_{0,2}(t,x)(\left(\frac{\partial}{\partial z}G\right)(y,z)v_2 b, \left(\frac{\partial^2}{\partial y\,\partial z}G\right)(y,z)(v_1,v_3)e_n) \\
&\quad + u_{0,2}(t,x)(\left(\frac{\partial}{\partial y}G\right)(y,z)v_3 b, \left(\frac{\partial^2}{\partial z^2}G\right)(y,z)(v_1,v_2)e_n) \\
&\quad + u_{0,2}(t,x)(G(y,z)e_n, \left(\frac{\partial^3}{\partial y\,\partial z^2}G\right)(y,z)(v_1,v_2v_3)e_n)\Bigg\|_V \\
&\leq 8c_{0,0}C_G^2 \max\{1,\|y\|_H,\|z\|_H\}^2. \qquad (2.131)
\end{aligned}$$

The fundamental theorem of calculus combined with (2.121)–(2.124) and (2.126)–(2.131) prove the second assertion (2.112).

Now we prove the third assertion. Let again $x, z \in H$ and $v_1, v_2, v_3 \in H$ with $\|v_1\|_H, \|v_2\|_H, \|v_3\|_H \leq 1$. Then we have

$$\begin{aligned}
\left\|\left(\frac{\partial}{\partial x}\phi\right)(x,z)v_1\right\|_V &= \left\|\left(\frac{\partial}{\partial x}\psi\right)(x,x,z)v_1 + \left(\frac{\partial}{\partial y}\psi\right)(x,x,z)v_1\right\|_V \\
&\leq 2\left(c_{0,0}+c_{0,0,0}\right)C_G^2\max\{1,\|x\|_H,\|z\|_H\}^2, \qquad (2.132)
\end{aligned}$$

$$\begin{aligned}
&\left\|\left(\frac{\partial^2}{\partial x^2}\phi\right)(x,z)(v_1,v_2)\right\|_V \\
&= \Bigg\|\left(\frac{\partial^2}{\partial x^2}\psi\right)(x,x,z)(v_1,v_2) + \left(\frac{\partial^2}{\partial x\,\partial y}\psi\right)(x,x,z)(v_2,v_1) \\
&\quad + \left(\frac{\partial^2}{\partial x\,\partial y}\psi\right)(x,x,z)(v_1,v_2) + \left(\frac{\partial^2}{\partial y^2}\psi\right)(x,x,z)(v_1,v_2)\Bigg\|_V \\
&\leq 4\left(c_{0,0}+c_{0,0,0}+c_{0,0,0,0}\right)C_G^2\max\{1,\|x\|_H,\|z\|_H\}^2, \qquad (2.133)
\end{aligned}$$

$$\begin{aligned}
&\left\| \left(\frac{\partial^2}{\partial x\, \partial z} \phi \right) (x,z)(v_1,v_2) \right\|_V \\
&\qquad = \left\| \left(\frac{\partial^2}{\partial x\, \partial z} \psi \right) (x,x,z)(v_1,v_2) + \left(\frac{\partial^2}{\partial y\, \partial z} \psi \right) (x,x,z)(v_1,v_2) \right\|_V \\
&\qquad \leq 4\, (c_{0,0} + c_{0,0,0})\, C_G^2 \max\{1, \|x\|_H, \|z\|_H\}^2, \qquad (2.134)
\end{aligned}$$

$$\begin{aligned}
&\left\| \left(\frac{\partial^3}{\partial x^2\, \partial z} \phi \right) (x,z)(v_1,v_2,v_3) \right\|_V \\
&\quad = \left\| \left(\frac{\partial^3}{\partial x^2\, \partial z} \psi \right) (x,x,z)(v_1,v_2,v_3) + \left(\frac{\partial^3}{\partial x\, \partial y\, \partial z} \psi \right) (x,x,z)(v_1,v_3,v_2) \right. \\
&\qquad \left. + \left(\frac{\partial^3}{\partial x\, \partial y\, \partial z} \psi \right) (x,x,z)(v_1,v_2,v_3) + \left(\frac{\partial^3}{\partial y^2\, \partial z} \psi \right) (x,x,z)(v_1,v_2,v_3) \right\|_V \\
&\quad \leq 8\, (c_{0,0} + c_{0,0,0} + c_{0,0,0,0})\, C_G^2 \max\{1, \|x\|_H, \|z\|_H\}^2, \qquad (2.135)
\end{aligned}$$

$$\begin{aligned}
&\left\| \left(\frac{\partial^3}{\partial x\, \partial z^2} \phi \right) (x,z)(v_1,v_2,v_3) \right\|_V \\
&\quad = \left\| \left(\frac{\partial^3}{\partial x\, \partial z^2} \psi \right) (x,x,z)(v_1,v_2,v_3) + \left(\frac{\partial^3}{\partial y\, \partial z^2} \psi \right) (x,x,z)(v_1,v_2,v_3) \right\|_V \\
&\quad \leq 8\, (c_{0,0} + c_{0,0,0})\, C_G^2 \max\{1, \|x\|_H, \|z\|_H\}^2. \qquad (2.136)
\end{aligned}$$

The last case is again handled differently. Observe that for all $x, x_1, x_2, z \in H$ and $v_1, v_2, v_3 \in H$ with $\|v_1\|_H, \|v_2\|_H, \|v_3\|_H \leq 1$ we have

$$\begin{aligned}
&\left\| \frac{\partial}{\partial x} \left[\left(\frac{\partial^2}{\partial x^2} \phi \right) (x,z) - \left(\frac{\partial^2}{\partial x^2} \psi \right) (x,x,z) \right] (v_1,v_2,v_3) \right\|_V \\
&\quad = \left\| \left(\frac{\partial^3}{\partial x^2\, \partial y} \psi \right) (x,x,z)(v_1,v_2,v_3) + \left(\frac{\partial^3}{\partial x^2\, \partial y} \psi \right) (x,x,z)(v_2,v_1,v_3) \right. \\
&\qquad + \left(\frac{\partial^3}{\partial x\, \partial y^2} \psi \right) (x,x,z)(v_1,v_2,v_3) + \left(\frac{\partial^3}{\partial x\, \partial y^2} \psi \right) (x,x,z)(v_1,v_3,v_2) \\
&\qquad \left. + \left(\frac{\partial^3}{\partial x\, \partial y^2} \psi \right) (x,x,z)(v_2,v_3,v_1) + \left(\frac{\partial^3}{\partial y^3} \psi \right) (x,x,z)(v_1,v_2,v_3) \right\|_V \\
&\quad \leq 12\, (c_{0,0} + c_{0,0,0} + c_{0,0,0,0})\, C_G^2 \max\{1, \|x\|_H, \|z\|_H\}^2 \qquad (2.137)
\end{aligned}$$

as well as

$$\begin{aligned}
&\left\|\left[\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_1,x_1,z)-\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_2,x_2,z)\right](v_1,v_2)\right\|_V\\
&\quad\le\left\|\left[\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_1,x_1,z)-\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_2,x_1,z)\right](v_1,v_2)\right\|_V\\
&\qquad+\left\|\left[\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_2,x_1,z)-\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_2,x_2,z)\right](v_1,v_2)\right\|_V\\
&\quad\le 2\left(c_{0,0,0,0}+\tilde{c}_{0,0,0,0}\right)C_G^2\max\{1,\|x_1\|_H,\|x_2\|_H,\|z\|_H\}^2\|x_1-x_2\|_H. \qquad (2.138)
\end{aligned}$$

Combining (2.137) and (2.138) with the fundamental theorem of calculus shows that for all $x_1,x_2,z\in H$ and $v_1,v_2\in H$ with $\|v_1\|_H,\|v_2\|_H\le 1$ we have

$$\begin{aligned}
&\left\|\left[\left(\frac{\partial^2}{\partial x^2}\phi\right)(x_1,z)-\left(\frac{\partial^2}{\partial x^2}\phi\right)(x_2,z)\right](v_1,v_2)\right\|_V\\
&\quad\le\left\|\left[\left(\frac{\partial^2}{\partial x^2}\phi\right)(x_1,z)-\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_1,x_1,z)\right](v_1,v_2)\right.\\
&\qquad\left.-\left[\left(\frac{\partial^2}{\partial x^2}\phi\right)(x_2,z)-\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_2,x_2,z)\right](v_1,v_2)\right\|_V\\
&\qquad+\left\|\left[\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_1,x_1,z)-\left(\frac{\partial^2}{\partial x^2}\psi\right)(x_2,x_2,z)\right](v_1,v_2)\right\|_V\\
&\quad\le 14\left(c_{0,0}+c_{0,0,0}+c_{0,0,0,0}+\tilde{c}_{0,0,0,0}\right)C_G^2\\
&\qquad\cdot\max\{1,\|x_1\|_H,\|x_2\|_H,\|z\|_H\}^2\|x_1-x_2\|_H. \qquad (2.139)
\end{aligned}$$

Finally, the fundamental theorem of calculus and (2.132)–(2.136) as well as (2.139) prove the last assertion (2.113). □

2.5 Weak Convergence for Regular Coefficients

Now we will combine the results from the previous sections. The idea is to rewrite the weak error in terms of the solution u of the Kolmogorov backward equation associated to the true solution X evaluated at suitable time points of the auxiliary process $\bar{Z}$ and then use the Itô formula Theorem 1.8. However, u is only differentiable on H_1. Thus, in order to apply the usual Itô formula we need $\bar{Z}$ to be a (usual) Itô process on H_1. That is why we now make the following stronger assumptions for this section.

Assumption 4 (Mollified coefficients).
Let $p\ge 2$. We assume that the initial value satisfies $\xi\in L^p(\Omega,H_2)$, that the drift coefficient satisfies $F\in \mathrm{Lip}^4(H,H_2)$ and that the diffusion coefficient satisfies $B\in \mathrm{Lip}^4(H,\mathrm{L}_2(U,H_2))$.

Assumption 5 (Mollified diffusion).
We assume that G satisfies the following assumptions:

i) *$G \in C^3(H \times H, \mathrm{L}_2(U, H_2))$,*

ii) *for all $x \in H$ we have $G(x, 0) = B(x)$ and*

iii) *there exists a constant $C_G \in [1, \infty)$ such that for all $x, y \in H$ and $0 \le i, j \le 3$ with $i + j \le 3$ we have*

$$\Big\| \Big(\frac{\partial^{i+j}}{\partial x^i \, \partial y^j} G \Big)(x, y) \Big\|_{L^{(i+j)}(H^{i+j}, \mathrm{L}_2(U, H_2))} \le C_G \max\{1, \|x\|_H, \|y\|_H\}.$$

Under these stronger assumptions we have $\bar{Z} \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H_1))$ and we will see in the following proof that also $\bar{Z} \in \mathfrak{C}^0_T(H_1)$. These assumptions also ensure the boundedness of the constants arising in Lemmas 2.6 and 2.7 — at least for $\vartheta \in [0, \frac{1}{2})$. In chapter 3, we will then see how to remove these extra smoothness assumptions to extend the results to more irregular coefficients.

Lemma 2.8.
Let $\vartheta \in [0, \frac{1}{2})$, $\rho \in [0, 1 - \vartheta)$, $p \ge 5$ and $\varphi \in \mathrm{Lip}^4(H, V)$. Then there exists a constant $C \in [0, \infty)$ such that

$$\Big\| \mathbb{E}\Big[\varphi(\bar{Z}_T) - \varphi(X_T) \Big] \Big\|_V \le C \cdot h^{\rho}.$$

Proof. Remember that the solution of the Kolmogorov backward equation is defined by $u(t, x) = \mathbb{E}[\varphi(X^x_{T-t})]$ for all $t \in [0, T]$, $x \in H$. This allows us to rewrite the weak distance solely in terms of u and $\bar{Z}$ as

$$\mathbb{E}\Big[\varphi(\bar{Z}_T) - \varphi(X_T) \Big] = \mathbb{E}\Big[u(T, \bar{Z}_T) - u(0, \bar{Z}_0) \Big]. \tag{2.140}$$

The next step will be to apply the standard Itô formula Theorem 1.8 to $\bar{Z}$ and u. Note that by Theorem 1.37 we have that $u \in C^{1,2}([0, T] \times H_1, V)$. Thus, we still need to check whether $\bar{Z} \in \mathfrak{C}^0_T(H_1)$. By Assumptions 4 and 5 we have for every $r \in [2, p]$ that

$$\begin{aligned} &\mathbb{E}\Big[\Big(\int_0^T \|F(Z_{\lfloor t \rfloor_h})\|_{H_2} \, \mathrm{d}t \Big)^r \Big]^{\frac{1}{r}} \\ &\quad \le \int_0^T \|F\|_{\mathrm{Lip}^0(H, H_2)} \|\max\{1, \|Z_{\lfloor t \rfloor_h}\|_H\}\|_{L^p(\Omega)} \, \mathrm{d}t \\ &\quad \le \|F\|_{\mathrm{Lip}^0(H, H_2)} T K_r \end{aligned} \tag{2.141}$$

as well as

$$\begin{aligned} &\mathbb{E}\Big[\Big(\int_0^T \|G(Z_{\lfloor t \rfloor_h}, V_{\lfloor t \rfloor_h, t})\|^2_{\mathrm{L}_2(U, H_2)} \, \mathrm{d}t \Big)^{\frac{r}{2}} \Big]^{\frac{1}{r}} \\ &\quad \le \left(\int_0^T C_G^2 \|\max\{1, \|Z_{\lfloor t \rfloor_h}\|_H, \|V_{\lfloor t \rfloor_h, t}\|_H\}\|^2_{L^p(\Omega)} \, \mathrm{d}t \right)^{\frac{1}{2}} \\ &\quad \le C_G (1 + C_{V,h,r}) T^{\frac{1}{2}} K_r. \end{aligned} \tag{2.142}$$

Therefore, we can employ Lemma 1.34 to get that $\bar{Z} \in \mathfrak{C}^r_T(H_1) \subseteq \mathfrak{C}^0_T(H_1)$ is a H_1-valued Itô process with

$$\bar{Z}_t = \xi + \int_0^t A\bar{Z}_s + F(Z_{\lfloor s \rfloor_h})\,\mathrm{d}s + \int_0^t G(Z_{\lfloor s \rfloor_h}, V_{\lfloor s \rfloor_h, s})\,\mathrm{d}W_s, \quad t \in [0,T] \tag{2.143}$$

and the following application of the Itô formula is valid. Remember the subscript notation $u_{i,j}$ which is used to denote the partial derivatives $\frac{\partial^{i+j}}{\partial t^i\,\partial x^j}u$. We get that

$$\begin{aligned}
&\mathbb{E}\Big[u(T,\bar{Z}_T) - u(0,\bar{Z}_0)\Big] \\
&\quad = \mathbb{E}\Bigg[\int_0^T u_{1,0}(t,\bar{Z}_t)\,\mathrm{d}t + \int_0^T u_{0,1}(t,\bar{Z}_t)\Big(A\bar{Z}_t + F(Z_{\lfloor t \rfloor_h})\Big)\,\mathrm{d}t \\
&\qquad + \int_0^T u_{0,1}(t,\bar{Z}_t)\Big(G(Z_{\lfloor t \rfloor_h}, V_{\lfloor t \rfloor_h, t})\Big)\,\mathrm{d}W_t \\
&\qquad + \frac{1}{2}\sum_{n\in\mathbb{N}}\int_0^T u_{0,2}(t,\bar{Z}_t)\Big(G(Z_{\lfloor t \rfloor_h}, V_{\lfloor t \rfloor_h, t})e_n, G(Z_{\lfloor t \rfloor_h}, V_{\lfloor t \rfloor_h, t})e_n\Big)\,\mathrm{d}t\Bigg].
\end{aligned} \tag{2.144}$$

Now we can see that the integrand of the stochastic integral lies in $\mathfrak{L}^{2,2}_T(\mathrm{L}_2(U,V))$ because

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \|u_{0,1}(t,\bar{Z}_t)\Big(G(Z_{\lfloor t \rfloor_h}, V_{\lfloor t \rfloor_h, t})\Big)\|^2_{\mathrm{L}_2(U,V)}\,\mathrm{d}t\right]^{\frac{1}{2}} \\
&\qquad \leq c_0\left(\int_0^T \|G(Z_{\lfloor t \rfloor_h}, V_{\lfloor t \rfloor_h, t})\|^2_{L^2(\Omega,\mathrm{L}_2(U,H))}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad \leq c_0 C_G(1 + C_{V,h,2})T^{\frac{1}{2}}K_2
\end{aligned} \tag{2.145}$$

which by Proposition 1.11 implies that the expectation of the stochastic integral in (2.144) vanishes. We can now use the fact, that u solves the Kolmogorov backward equation (1.13) associated to X to substitute the time derivative of u by space derivatives which can then be compared to the existing space derivatives introduced by the Itô formula. This allows us to then use the boundedness and Lipschitz estimates in Theorem 1.37 to establish the weak convergence rate. We obtain that

$$\begin{aligned}
&\mathbb{E}\Big[u(T,\bar{Z}_T) - u(0,\bar{Z}_0)\Big] \\
&\quad = \mathbb{E}\Bigg[\int_0^T u_{0,1}(t,\bar{Z}_t)\Big(F(Z_{\lfloor t \rfloor_h})\Big) - u_{0,1}(t,\bar{Z}_t)\Big(F(\bar{Z}_t)\Big)\,\mathrm{d}t \\
&\qquad + \frac{1}{2}\sum_{n\in\mathbb{N}}\int_0^T u_{0,2}(t,\bar{Z}_t)\Big(G(Z_{\lfloor t \rfloor_h}, V_{\lfloor t \rfloor_h, t})e_n, G(Z_{\lfloor t \rfloor_h}, V_{\lfloor t \rfloor_h, t})e_n\Big) \\
&\qquad - u_{0,2}(t,\bar{Z}_t)\Big(B(\bar{Z}_t)e_n, B(\bar{Z}_t)e_n\Big)\,\mathrm{d}t\Bigg].
\end{aligned} \tag{2.146}$$

Next we add and subtract some terms and use the triangle inequality in such a way that we can make use of Propositions 2.4 and 2.5. Specifically, we get

$$\begin{align}
&\Big\|\mathbb{E}\Big[u(T,\bar{Z}_T)-u(0,\bar{Z}_0)\Big]\Big\|_V \tag{2.147}\\
&\leq \int_0^T \Big\|\mathbb{E}\Big[u_{0,1}(t,\bar{Z}_t)\Big(F(Z_{\lfloor t\rfloor_h})\Big)-u_{0,1}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(F(Z_{\lfloor t\rfloor_h})\Big)\Big]\Big\|_V \,\mathrm{d}t \notag\\
&\quad+\int_0^T \Big\|\mathbb{E}\Big[u_{0,1}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(F(Z_{\lfloor t\rfloor_h})\Big)-u_{0,1}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(F(\bar{Z}_{\lfloor t\rfloor_h})\Big)\Big]\Big\|_V \,\mathrm{d}t \notag\\
&\quad+\int_0^T \Big\|\mathbb{E}\Big[u_{0,1}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(F(\bar{Z}_{\lfloor t\rfloor_h})\Big)-u_{0,1}(t,\bar{Z}_t)\Big(F(\bar{Z}_t)\Big)\Big]\Big\|_V \,\mathrm{d}t \notag\\
&\quad+\frac{1}{2}\int_0^T \Big\|\mathbb{E}\Big[\sum_{n\in\mathbb{N}} u_{0,2}(t,\bar{Z}_t)\Big(G(Z_{\lfloor t\rfloor_h},V_{\lfloor t\rfloor_h,t})e_n,G(Z_{\lfloor t\rfloor_h},V_{\lfloor t\rfloor_h,t})e_n\Big) \notag\\
&\quad-u_{0,2}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(B(Z_{\lfloor t\rfloor_h})e_n,B(Z_{\lfloor t\rfloor_h})e_n\Big)\Big]\Big\|_V \,\mathrm{d}t \notag\\
&\quad+\frac{1}{2}\int_0^T \Big\|\mathbb{E}\Big[\sum_{n\in\mathbb{N}} u_{0,2}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(B(Z_{\lfloor t\rfloor_h})e_n,B(Z_{\lfloor t\rfloor_h})e_n\Big) \notag\\
&\quad-u_{0,2}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(B(\bar{Z}_{\lfloor t\rfloor_h})e_n,B(\bar{Z}_{\lfloor t\rfloor_h})e_n\Big)\Big]\Big\|_V \,\mathrm{d}t \notag\\
&\quad+\frac{1}{2}\int_0^T \Big\|\mathbb{E}\Big[\sum_{n\in\mathbb{N}} u_{0,2}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(B(\bar{Z}_{\lfloor t\rfloor_h})e_n,B(\bar{Z}_{\lfloor t\rfloor_h})e_n\Big) \notag\\
&\quad-u_{0,2}(t,\bar{Z}_t)\Big(B(\bar{Z}_t)e_n,B(\bar{Z}_t)e_n\Big)\Big]\Big\|_V \,\mathrm{d}t. \tag{2.148}
\end{align}$$

Now we use Proposition 2.4 together with Lemmas 2.6 and 2.7 to get that there exists a constant $L_1\in[0,\infty)$ such that it holds for all $t\in(0,T)$

$$\begin{aligned}
&\Big\|\mathbb{E}\Big[u_{0,1}(t,\bar{Z}_t)\Big(F(Z_{\lfloor t\rfloor_h})\Big)-u_{0,1}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(F(Z_{\lfloor t\rfloor_h})\Big)\Big]\Big\|_V\\
&\quad+\Big\|\mathbb{E}\Big[u_{0,1}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(F(\bar{Z}_{\lfloor t\rfloor_h})\Big)-u_{0,1}(t,\bar{Z}_t)\Big(F(\bar{Z}_t)\Big)\Big]\Big\|_V\\
&\quad+\frac{1}{2}\Big\|\mathbb{E}\Big[\sum_{n\in\mathbb{N}} u_{0,2}(t,\bar{Z}_t)\Big(G(Z_{\lfloor t\rfloor_h},V_{\lfloor t\rfloor_h,t})e_n,G(Z_{\lfloor t\rfloor_h},V_{\lfloor t\rfloor_h,t})e_n\Big)\\
&\quad-u_{0,2}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(B(Z_{\lfloor t\rfloor_h})e_n,B(Z_{\lfloor t\rfloor_h})e_n\Big)\Big]\Big\|_V\\
&\quad+\frac{1}{2}\Big\|\mathbb{E}\Big[\sum_{n\in\mathbb{N}} u_{0,2}(t,\bar{Z}_{\lfloor t\rfloor_h})\Big(B(\bar{Z}_{\lfloor t\rfloor_h})e_n,B(\bar{Z}_{\lfloor t\rfloor_h})e_n\Big)\\
&\quad-u_{0,2}(t,\bar{Z}_t)\Big(B(\bar{Z}_t)e_n,B(\bar{Z}_t)e_n\Big)\Big]\Big\|_V\\
&\leq L_1\cdot\max\{1,\frac{1}{(T-t)^{\vartheta}}\}\cdot\frac{(t-\lfloor t\rfloor_h)^{\rho}}{t^{\rho}}\,.
\end{aligned}\tag{2.149}$$

In particular, we can choose

$$
\begin{aligned}
L_1 = 72\,&[c_\vartheta + c_{\vartheta,0} + c_{\vartheta,0,0} + c_{\vartheta,0,0,0} + c_{0,0} + c_{0,0,0} + c_{0,0,0,0} + \tilde{c}_{0,0,0,0}] \\
&\cdot C_0^4 C_\rho^2 C_{\rho+\vartheta} C_\vartheta K_5^5 (1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}) \qquad (2.150)\\
&\cdot \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^2 \cdot \max\{1, \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}, C_G(1 + C_{V,T,5})\}^6.
\end{aligned}
$$

Note that here we need the assumption that $p \geq 5$, since, on the one hand, by Lemma 2.7 the function $(x, y, z) \mapsto \sum_{n\in\mathbb{N}} u_{0,2}(t, x)(G(y, z)e_n, G(y, z)e_n)$ fulfils the assumption of Proposition 2.4 with $q = 2$ but, on the other hand, we required $q \in [0, p-3]$ to ensure the constant is finite. Thus, we need to have at least the first five moments finite. Similarly, an application of Proposition 2.5 together with Lemmas 2.6 and 2.7 shows that there exists a constant $L_2 \in [0, \infty)$ such that it holds for all $t \in [h, T)$

$$
\begin{aligned}
&\Big\|\mathbb{E}\Big[u_{0,1}(t, \bar{Z}_{\lfloor t\rfloor_h})\Big(F(Z_{\lfloor t\rfloor_h})\Big) - u_{0,1}(t, \bar{Z}_{\lfloor t\rfloor_h})\Big(F(\bar{Z}_{\lfloor t\rfloor_h})\Big)\Big]\Big\|_V \\
&\quad + \frac{1}{2}\Big\|\mathbb{E}\Big[\sum_{n\in\mathbb{N}} u_{0,2}(t, \bar{Z}_{\lfloor t\rfloor_h})\Big(B(Z_{\lfloor t\rfloor_h})e_n, B(Z_{\lfloor t\rfloor_h})e_n\Big) \\
&\quad - u_{0,2}(t, \bar{Z}_{\lfloor t\rfloor_h})\Big(B(\bar{Z}_{\lfloor t\rfloor_h})e_n, B(\bar{Z}_{\lfloor t\rfloor_h})e_n\Big)\Big]\Big\|_V \\
&\leq L_2 \cdot \max\{1, \frac{1}{(T-t)^\vartheta}\} \cdot \frac{h^\rho}{\lfloor t\rfloor_h^\rho} \qquad (2.151)\\
&\leq 2^\rho L_2 \cdot \max\{1, \frac{1}{(T-t)^\vartheta}\} \cdot \frac{h^\rho}{t^\rho} \qquad (2.152)
\end{aligned}
$$

while for $t \in (0, h)$ both terms vanish. For the constant we can choose

$$
\begin{aligned}
L_2 = 8\,&[c_\vartheta + c_{\vartheta,0} + c_{\vartheta,0,0} + c_{0,0} + c_{0,0,0} + c_{0,0,0,0}] \\
&\cdot C_0^4\,[C_{0,0,\rho} + C_{\vartheta,0,\rho} + C_\vartheta C_{0,\rho} + C_\vartheta C_\rho A_{T,\rho,5}]\, K_5^5 (1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}) \qquad (2.153)\\
&\cdot \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^2 \cdot \max\{1, \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}, C_G(1 + C_{V,T,5})\}^4.
\end{aligned}
$$

This combined with the fact that $t - \lfloor t\rfloor_h < h$ proofs that

$$
\begin{aligned}
\Big\|\mathbb{E}\Big[\varphi(\bar{Z}_T) - \varphi(X_T)\Big]\Big\|_V &\leq \int_0^T (L_1 + 2^\rho L_2) \cdot \max\left\{1, \frac{1}{(T-t)^\vartheta}\right\} \cdot \frac{h^\rho}{t^\rho}\,\mathrm{d}t \qquad (2.154)\\
&\leq (L_1 + 2L_2) \cdot \left(\frac{T^{1-\rho}}{1-\rho} + \frac{2T^{1-\vartheta-\rho}}{1-\vartheta-\rho}\right) \cdot h^\rho, \qquad (2.155)
\end{aligned}
$$

where we used that

$$
\int_0^T \max\left\{1, \frac{1}{(T-t)^\vartheta}\right\} \cdot \frac{1}{t^\rho}\,\mathrm{d}t \leq \int_0^T \frac{1}{t^\rho}\,\mathrm{d}t + \int_0^T \frac{1}{(T-t)^\vartheta} \cdot \frac{1}{t^\rho}\,\mathrm{d}t \qquad (2.156)
$$

and

$$\begin{aligned}
\int_0^T \frac{1}{(T-t)^{\vartheta}} \cdot \frac{1}{t^{\rho}} \,\mathrm{d}t &\leq \int_0^T \max\left\{\frac{1}{T-t}, \frac{1}{t}\right\}^{\vartheta+\rho} \mathrm{d}t && (2.157)\\
&\leq \int_0^T (T-t)^{-(\vartheta+\rho)} \,\mathrm{d}t + \int_0^T t^{-(\vartheta+\rho)} \,\mathrm{d}t && (2.158)\\
&= \frac{2T^{1-\vartheta-\rho}}{1-\vartheta-\rho}. && (2.159)
\end{aligned}$$

For the overall constant we arrive at

$$\begin{aligned}
C &= (L_1 + 2L_2) \cdot \left(\frac{T^{1-\rho}}{1-\rho} + \frac{2T^{1-\vartheta-\rho}}{1-\vartheta-\rho}\right) && (2.160)\\
&\leq 144 \left[c_{\vartheta} + c_{\vartheta,0} + c_{\vartheta,0,0} + c_{\vartheta,0,0,0} + c_{0,0} + c_{0,0,0} + c_{0,0,0,0} + \tilde{c}_{0,0,0,0}\right]\\
&\quad \cdot C_0^4 \left[C_{\rho}^2 C_{\rho+\vartheta} C_{\vartheta} + C_{0,0,\rho} + C_{\vartheta,0,\rho} + C_{\vartheta} C_{0,\rho} + C_{\vartheta} C_{\rho} A_{T,\rho,5}\right]\\
&\quad \cdot \max\left\{1, \frac{1}{T}\right\} \cdot \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right)^2 \cdot K_5^5 && (2.161)\\
&\quad \cdot \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^2 \cdot \max\{1, \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}, C_G(1 + C_{V,T,5})\}^6
\end{aligned}$$

□

Lemma 2.9.
Let $\vartheta \in [0,1)$, $\rho \in [0, 1-\vartheta)$, $p \geq 3$ and $\varphi \in \mathrm{Lip}^4(H,V)$. Then there exists a constant $C \in [0,\infty)$ such that

$$\left\|\mathbb{E}\Big[\varphi(Z_T) - \varphi(\bar{Z}_T)\Big]\right\|_V \leq C \cdot h^{\rho}.$$

Proof. Since $\varphi \in \mathrm{Lip}^4(H,V)$ we have in particular

$$\max_{i\in\{0,1,2\}} \left\|\left(\frac{\partial^i}{\partial x^i}\varphi\right)(x) - \left(\frac{\partial^i}{\partial x^i}\varphi\right)(y)\right\|_{L^{(i)}(H,V)} \leq \|\varphi\|_{\mathrm{Lip}^2(H,V)} \|x-y\|_H \tag{2.162}$$

for all $x, y \in H$. This means that we can apply Proposition 2.5 with $\psi(x,y,z) = \varphi(y)$ to get that there exists a constant $C \in [0,\infty)$ such that

$$\left\|\mathbb{E}\Big[\varphi(Z_T) - \varphi(\bar{Z}_T)\Big]\right\|_V \leq \frac{C}{T^{\rho}} \cdot h^{\rho}. \tag{2.163}$$

The constant C can be chosen as

$$\begin{aligned}
C &= 2\|\varphi\|_{\mathrm{Lip}^2(H,V)}\\
&\quad \cdot C_0^2 \left[C_{0,0,\rho} + C_{\vartheta,0,\rho} + C_{\vartheta} C_{0,\rho} + C_{\vartheta} C_{\rho} A_{T,\rho,3}\right]\\
&\quad \cdot \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right)^2 \cdot K_3^3\\
&\quad \cdot \max\{1, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\} \cdot \max\{1, \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}, C_G(1 + C_{V,T,3})\}^2 && (2.164)
\end{aligned}$$

□

Combining Lemmas 2.8 and 2.9 with the triangle inequality we get the following weak convergence result for the smooth case of Assumptions 4 and 5.

Corollary 2.10.
Let $\vartheta \in [0, \frac{1}{2})$*,* $\rho \in [0, 1-\vartheta)$*,* $p \geq 5$ *and* $\varphi \in \mathrm{Lip}^4(H, V)$*. Then there exists a constant* $C \in [0, \infty)$ *such that*

$$\|\mathbb{E}[\varphi(Z_T) - \varphi(X_T)]\|_V \leq C \cdot h^\rho.$$

Proof. Using the triangle inequality we see that we can choose

$$\begin{aligned} C = 144 &\left[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + c_\vartheta + c_{\vartheta,0} + c_{\vartheta,0,0} + c_{\vartheta,0,0,0} + c_{0,0} + c_{0,0,0} + c_{0,0,0,0} + \tilde{c}_{0,0,0,0}\right] \\ &\cdot C_0^4 \left[C_\rho^2 C_{\rho+\vartheta} C_\vartheta + C_{0,0,\rho} + C_{\vartheta,0,\rho} + C_\vartheta C_{0,\rho} + C_\vartheta C_\rho A_{T,\rho,5}\right] \\ &\cdot \max\left\{1, \frac{1}{T}\right\} \cdot \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right)^2 \cdot K_5^5 \\ &\cdot \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^2 \cdot \max\{1, \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}, C_G(1 + C_{V,T,5})\}^6. \end{aligned}$$

We can estimate further using the estimate of $A_{T,\rho,5}$ (2.19) and the definition of $C_{V,T,5}$ (see Lemma 2.1) as

$$\begin{aligned} C \leq 288 &\left[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + c_\vartheta + c_{\vartheta,0} + c_{\vartheta,0,0} + c_{\vartheta,0,0,0} + c_{0,0} + c_{0,0,0} + c_{0,0,0,0} + \tilde{c}_{0,0,0,0}\right] \\ &\cdot C_0^4 \left[C_\rho^2 C_{\rho+\vartheta} C_\vartheta + C_{0,0,\rho} + C_{\vartheta,0,\rho} + C_\vartheta C_{0,\rho} + C_\vartheta C_\rho (C_{\rho,\rho} + C_{\vartheta,-\rho,\rho} + \tau_5 C_{0,-\rho,\rho})\right] \\ &\cdot \max\left\{1, \frac{1}{T}\right\} \cdot \left(1 + \tau_5 T^{1/2}\right)^7 \cdot \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right)^3 \cdot K_5^5 \\ &\cdot C_G^7 \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^3 \cdot \max\{1, \|B\|_{\mathrm{Lip}^0(H,\mathrm{L}_2(U,H))}\}^7. \end{aligned} \tag{2.165}$$

□

Chapter 3

Weak Convergence for Irregular Coefficients

Now that we have seen how to approximate the solution of very regular equations we come back to the general setting and describe a mollification approach that allows to transfer the weak convergence results from chapter 2 to irregular coefficients. This mollification procedure is based on similar approaches in [CJK19; JK21] and exploits the excellent smoothing properties of the analytic semigroup. Unfortunately, the combination of this mollification and the techniques used in chapter 2 do not allow us to obtain the expected optimal order of convergence in all cases, namely we lose something for highly irregular drift functions with values in the spaces $H_{-\vartheta}$ for $\vartheta \geq \frac{1}{2}$.

This chapter is structured as follows. We start section 3.1 with a brief detour into the theory of fractional integral equations to motivate a generalised version of Grönwall's lemma. Afterwards, we recall a useful perturbation estimate based on this Grönwall's lemma and the Burkholder–Davis–Gundy inequality. Next, we explain in detail the mollification procedure in section 3.2. The proofs that this strategy is indeed well-behaved all rely on the aforementioned perturbation estimate. Finally, after the preparation is done we will finish the main weak convergence proof in section 3.3.

In this chapter we again assume the general setting detailed at the start of chapter 2.

3.1 A Perturbation Estimate

In the following, we need a generalisation of the exponential function called the Mittag-Leffler function (named after the Swedish mathematician Gösta Mittag-Leffler). It plays a similar role as the exponential function but in the context of the fractional calculus and in particular the theory of fractional differential equations [Die10; Gor+14]. It is defined for $\alpha \in (0, \infty)$ and $x \in \mathbb{R}$ as an infinite series

$$\mathrm{E}_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \tag{3.1}$$

where Γ denotes the Gamma function. As the special case $\alpha = 1$ we obtain $\mathrm{E}_1(x) = e^x$. Note that $\mathrm{E}_\alpha(x) \geq 1$ for all $x \geq 0$ and $\alpha \in (0, \infty)$.

Using the Mittag-Leffler function one can formulate a kind of "fractional" Grönwall's lemma, see e.g. [Hen81, Section 7.1] and [Yag10, Section 1.9.1]. A common form of Grönwall's lemma states that all functions satisfying a particular integral inequality are bounded pointwise by the solution of the corresponding integral equation. Let us look at a simple example. Suppose we have a continuous function $u \in C([0,T],\mathbb{R})$ satisfying

$$u(t) \le 1 + \int_0^t u(s)\,\mathrm{d}s \quad \forall t \in [0,T]. \tag{3.2}$$

The corresponding integral equation is given by

$$u^*(t) = 1 + \int_0^t u^*(s)\,\mathrm{d}s \quad \forall t \in [0,T] \tag{3.3}$$

for which the solution is readily seen to be $u^*(t) = e^t$ for all $t \in [0,T]$. Thus, Grönwall's lemma states that we can bound the original function as

$$u(t) \le e^t \quad \forall t \in [0,T]. \tag{3.4}$$

We now want to generalise this statement to consider also fractional integrals in (3.2) and (3.3). In particular, we will use the so-called Riemann–Liouville integral of order α which is defined for $\alpha > 0$ and continuous functions $u \in C([0,T],\mathbb{R})$ as the continuous function $\mathrm{B}_\alpha\, u \in C([0,T],\mathbb{R})$ given by

$$(\mathrm{B}_\alpha\, u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,\mathrm{d}s \quad \forall t \in [0,T]. \tag{3.5}$$

This definition is motivated by Cauchy's formula for repeated integrals, by which we already see that B_n for an integer $n \in \mathbb{N}$ coincides with the usual nth integral of a function. In particular, for $n = 1$ we have the usual integral $(\mathrm{B}_1\, u)(t) = \int_0^t u(s)\,\mathrm{d}s$. Furthermore, utilising the Euler integral $\int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,\mathrm{d}t = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$ one can show that these fractional integral operators satisfy a semigroup property, i.e. for all $\alpha, \beta > 0$ it holds $\mathrm{B}_\alpha\, \mathrm{B}_\beta = \mathrm{B}_{\alpha+\beta}$.

If we now consider the fractional analogue of the integral equation (3.3) for $\alpha > 0$

$$u^*(t) = 1 + (\mathrm{B}_\alpha\, u^*)(t) \quad \forall t \in [0,T] \tag{3.6}$$

then the solution of this equation can be expressed in terms of the Mittag-Leffler function as $u^*(t) = \mathrm{E}_\alpha(t^\alpha)$. This underscores the importance of the Mittag-Leffler function as the fractional analogue of the classical exponential function. The fractional version of Grönwall's lemma now tells us that, analogously to the classical case, every function $u \in C([0,T],\mathbb{R})$ satisfying the integral inequality

$$u(t) \le 1 + (\mathrm{B}_\alpha\, u)(t) \quad \forall t \in [0,T] \tag{3.7}$$

can be bounded as $u(t) \le \mathrm{E}_\alpha(t^\alpha)$ for all $t \in [0,T]$.

Theorem 3.1 (Fractional Grönwall's lemma).
Let $a, b \ge 0$ and $\alpha > 0$. If $u \in C([0,T],\mathbb{R})$ satisfies the integral inequality

$$u(t) \le a + \frac{b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,\mathrm{d}s \quad \forall t \in [0,T]$$

then the following bound holds

$$u(t) \le a\,\mathrm{E}_\alpha(bt^\alpha) \quad \forall t \in [0,T].$$

For the proof we refer to [Yag10, Theorem 1.26] or [Hen81, Lemma 7.1.1].

With the help of the Burkholder–Davis–Gundy inequality (Corollary 1.10) and this generalised Grönwall's lemma one can then establish the following classical perturbation estimate. A proof of the following proposition can be found in [JK21, Corollary 3.1].

Proposition 3.2.
Let $r \geq 2$, $\vartheta \in [0,1)$ *and* $X, \bar{X} \in \mathbb{M}(\mathcal{P}_T, \mathcal{B}(H))$ *be two stochastic processes with* $\sup_{t\in[0,T]} \left\| X_t - \bar{X}_t \right\|_{L^r(\Omega,H)} < \infty$. *If there exist* $y, z \in [0,\infty)$ *such that for every* $t \in (0,T]$ *there are processes* $\varphi^t, \bar{\varphi}^t \in \mathfrak{L}_t^{0,1}(H)$ *and* $\Phi^t, \bar{\Phi}^t \in \mathfrak{L}_T^{0,2}(\mathrm{L}_2(U,H))$ *such that for all* $s \in (0,t)$ *it holds that*

$$\left\| \varphi_s^t - \bar{\varphi}_s^t \right\|_{L^r(\Omega,H)} \leq y \cdot \frac{\sup_{u\in[0,s]} \left\| X_u - \bar{X}_u \right\|_{L^r(\Omega,H)}}{(t-s)^{\vartheta}}$$

$$\left\| \Phi_s^t - \bar{\Phi}_s^t \right\|_{L^r(\Omega,\mathrm{L}_2(U,H))} \leq z \cdot \frac{\sup_{u\in[0,s]} \left\| X_u - \bar{X}_u \right\|_{L^r(\Omega,H)}}{(t-s)^{\frac{\vartheta}{2}}}$$

then we have the explicit bound

$$\begin{aligned} &\sup_{t\in[0,T]} \left\| X_t - \bar{X}_t \right\|_{L^r(\Omega,H)} \\ &\quad \leq \sqrt{2} \cdot \mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta) \left(y \frac{T^{1-\vartheta}}{1-\vartheta} + \tau_r z \sqrt{\frac{T^{1-\vartheta}}{1-\vartheta}} \right)^2 \right)^{\frac{1}{2}} \\ &\qquad \cdot \sup_{t\in[0,T]} \left\| X_t - \int_0^t \varphi_s^t \,\mathrm{d}s - \int_0^t \Phi_s^t \,\mathrm{d}W_s - \left[\bar{X}_t - \int_0^t \bar{\varphi}_s^t \,\mathrm{d}s - \int_0^t \bar{\Phi}_s^t \,\mathrm{d}W_s \right] \right\|_{L^r(\Omega,H)} \\ &\quad < \infty. \end{aligned}$$

3.2 A Mollification Procedure

Let us now describe the mollification procedure. As we have seen in section 1.2.1 (especially Theorem 1.24) the analytic semigroup generated by the dominating linear operator provides a strong regularisation effect. We make use of this regularisation to mollify the coefficients including the initial value of the SEE. We define for $\varepsilon \in [0,T]$ the mollified coefficients

$$F_\varepsilon \colon H \to H_{-\vartheta}, \quad x \mapsto e^{\varepsilon A} F(x) \tag{3.8}$$

$$B_\varepsilon \colon H \to \mathrm{L}_2(U,H), \quad x \mapsto e^{\varepsilon A} B(x) \tag{3.9}$$

$$G_\varepsilon \colon H \times H \to \mathrm{L}_2(U,H), \quad (x,y) \mapsto e^{\varepsilon A} G(x,y) \tag{3.10}$$

and separately the mollified initial value for $\delta \in [0,T]$

$$\xi_\delta \in L^p(\Omega,H), \quad \xi_\delta = e^{\delta A} \xi. \tag{3.11}$$

Note that, thanks to the regularisation the coefficients F_ε, B_ε and G_ε actually take values in H_α for every $\alpha \in \mathbb{R}$ for strictly positive $\varepsilon \in (0,T]$. In particular this means, that we have $F_\varepsilon \in \operatorname{Lip}^4(H, H_2)$ and $B_\varepsilon \in \operatorname{Lip}^4(H, \mathrm{L}_2(U, H_2))$ for $\varepsilon \in (0,T]$. Similarly we have $\xi_\delta \in L^p(\Omega, H_\alpha)$ for every $\alpha \in \mathbb{R}$ in case $\delta \in (0,T]$. Therefore these mollified coefficients satisfy Assumptions 4 and 5. In particular, the latter one is satisfied with constant $C_{G_\varepsilon} = C_0 C_G$.

Now we look at the mollified evolution equation

$$\begin{cases} \mathrm{d}X_t^{\varepsilon,\delta} = \Big[AX_t^{\varepsilon,\delta} + F_\varepsilon(X_t^{\varepsilon,\delta})\Big]\,\mathrm{d}t + B_\varepsilon(X_t^{\varepsilon,\delta})\,\mathrm{d}W_t, \quad t \in (0,T], \\ X_0^{\varepsilon,\delta} = \xi_\delta \end{cases} \tag{3.12}$$

which by Theorem 1.28 has a mild solution $X^{\varepsilon,\delta}$ given by $X_0^{\varepsilon,\delta} = \xi_\delta$ and

$$X_t^{\varepsilon,\delta} = e^{tA}X_0^{\varepsilon,\delta} + \int_0^t e^{(t-s)A}F_\varepsilon(X_s^{\varepsilon,\delta})\,\mathrm{d}s + \int_0^t e^{(t-s)A}B_\varepsilon(X_s^{\varepsilon,\delta})\,\mathrm{d}W_s \tag{3.13}$$

that satisfies $\sup_{t\in[0,T]}\|X_t^{\varepsilon,\delta}\|_{L^p(\Omega,H)} < \infty$ for every $\varepsilon, \delta \in [0,T]$. Additionally, we define the corresponding approximation $Z^{\varepsilon,\delta}$ by $Z_0^{\varepsilon,\delta} = \xi_\delta$ and

$$Z_t^{\varepsilon,\delta} = S_{0,t}Z_0^{\varepsilon,\delta} + \int_0^t S_{s,t}R_sF_\varepsilon(Z_{\lfloor s\rfloor_h}^{\varepsilon,\delta})\,\mathrm{d}s + \int_0^t S_{s,t}R_sG_\varepsilon\Big(Z_{\lfloor s\rfloor_h}^{\varepsilon,\delta}, \int_{\lfloor s\rfloor_h}^s B(Z_{\lfloor s\rfloor_h}^{\varepsilon,\delta})\,\mathrm{d}W_u\Big)\,\mathrm{d}W_s. \tag{3.14}$$

We also have $\sup_{t\in[0,T]}\|Z_t^{\varepsilon,\delta}\|_{L^p(\Omega,H)} < \infty$ for every $\varepsilon, \delta \in [0,T]$ by Lemma 2.2 since Assumption 3 is fulfilled.

3.2.1 Explicit Moment Bounds

We now use Proposition 3.2 to obtain a priori estimates for both $X^{0,\delta}$ and $Z^{0,\delta}$. This will then also allow us to provide a quantitative version of Lemma 2.2.

Lemma 3.3.
Let $r \geq 2$ and $\delta \in [0,T]$. Suppose that $\sup_{t\in[0,T]}\|X_t^{0,\delta}\|_{L^r(\Omega,H)} < \infty$. Then we have

$$\begin{aligned} &\sup_{t\in[0,T]}\|X_t^{0,\delta}\|_{L^r(\Omega,H)} \\ &\quad\leq \sqrt{2}\left[C_0\|\xi_\delta\|_{L^r(\Omega,H)} + C_\vartheta\|F(0)\|_{H_{-\vartheta}}\frac{T^{1-\vartheta}}{1-\vartheta} + \tau_r C_0\|B(0)\|_{\mathrm{L}_2(U,H)}T^{\frac{1}{2}}\right] \\ &\quad\cdot \mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta)\left(C_\vartheta|F|_{\operatorname{Lip}(H,H_{-\vartheta})}\frac{T^{1-\vartheta}}{1-\vartheta} + \tau_r C_0|B|_{\operatorname{Lip}(H,\mathrm{L}_2(U,H))}T\sqrt{\frac{T^{1-\vartheta}}{1-\vartheta}}\right)^2\right)^{\frac{1}{2}} \end{aligned}$$

Proof. We use Proposition 3.2 with

$$\begin{aligned} X_t &= X_t^{0,\delta}, & \varphi_s^t &= e^{(t-s)A}F(X_s^{0,\delta}), & \Phi_s^t &= e^{(t-s)A}B(X_s^{0,\delta}), \\ \bar{X}_t &= 0, & \bar{\varphi}_s^t &= e^{(t-s)A}F(0) \quad \text{and} & \bar{\Phi}_s^t &= e^{(t-s)A}B(0). \end{aligned}$$

By assumption, we have $\sup_{t\in[0,T]}\|X_t-\bar{X}_t\|_{L^r(\Omega,H)}<\infty$. Furthermore, we get that

$$\left\|\varphi_s^t-\bar{\varphi}_s^t\right\|_{L^r(\Omega,H)} \le C_\vartheta (t-s)^{-\vartheta}|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\|X_s^{0,\delta}\|_{L^r(\Omega,H)} \tag{3.15}$$

$$\le C_\vartheta|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{\sup_{u\in[0,s]}\|X_u^{0,\delta}\|_{L^r(\Omega,H)}}{(t-s)^\vartheta} \tag{3.16}$$

as well as

$$\left\|\Phi_s^t-\bar{\Phi}_s^t\right\|_{L^r(\Omega,\mathrm{L}_2(U,H))} \le C_0|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\|X_s^{0,\delta}\|_{L^r(\Omega,H)} \tag{3.17}$$

$$\le C_0|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}T\frac{\sup_{u\in[0,s]}\|X_u^{0,\delta}\|_{L^r(\Omega,H)}}{(t-s)^{\frac{\vartheta}{2}}}. \tag{3.18}$$

Proposition 3.2 thus implies that

$$\begin{aligned}\sup_{t\in[0,T]}&\|X_t^{0,\delta}\|_{L^r(\Omega,H)}\\ &\le\sqrt{2}\cdot\sup_{t\in[0,T]}\left\|e^{tA}X_0^{0,\delta}+\int_0^t e^{(t-s)A}F(0)\,\mathrm{d}s+\int_0^t e^{(t-s)A}B(0)\,\mathrm{d}W_s\right\|_{L^r(\Omega,H)}\\ &\cdot\mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta)\left(C_\vartheta|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{T^{1-\vartheta}}{1-\vartheta}+\tau_rC_0|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}T\sqrt{\frac{T^{1-\vartheta}}{1-\vartheta}}\right)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.19}$$

Using the triangle inequality and further estimating the terms in the supremum as

$$\sup_{t\in[0,T]}\left\|e^{tA}X_0^{0,\delta}\right\|_{L^r(\Omega,H)}\le C_0\|\xi_\delta\|_{L^r(\Omega,H)}, \tag{3.20}$$

$$\sup_{t\in[0,T]}\left\|\int_0^t e^{(t-s)A}F(0)\,\mathrm{d}s\right\|_{L^r(\Omega,H)}\le\|F(0)\|_{H_{-\vartheta}}C_\vartheta\frac{T^{1-\vartheta}}{1-\vartheta} \tag{3.21}$$

and

$$\sup_{t\in[0,T]}\left\|\int_0^t e^{(t-s)A}B(0)\,\mathrm{d}W_s\right\|_{L^r(\Omega,H)}\le\tau_r\|B(0)\|_{\mathrm{L}_2(U,H)}C_0T^{\frac{1}{2}} \tag{3.22}$$

completes the proof. □

Lemma 3.4.
Let $r\ge 2$ and $\delta\in[0,T]$. Suppose that $\sup_{t\in[0,T]}\|Z_t^{0,\delta}\|_{L^r(\Omega,H)}<\infty$. Then we have

$$\begin{aligned}\sup_{t\in[0,T]}&\|Z_t^{0,\delta}\|_{L^r(\Omega,H)}\\ &\le\sqrt{2}\left[C_0\|\xi_\delta\|_{L^r(\Omega,H)}+C_\vartheta\|F(0)\|_{H_{-\vartheta}}\frac{T^{1-\vartheta}}{1-\vartheta}+\tau_rC_0C_G(1+\tau_r\|B(0)\|_{\mathrm{L}_2(U,H)}h^{\frac{1}{2}})T^{\frac{1}{2}}\right]\\ &\cdot\mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta)\left(C_\vartheta|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{T^{1-\vartheta}}{1-\vartheta}+\tau_rC_0C_G(1+\tau_rh^{\frac{1}{2}})T\sqrt{\frac{T^{1-\vartheta}}{1-\vartheta}}\right)^2\right)^{\frac{1}{2}}.\end{aligned}$$

Proof. We use Proposition 3.2 with

$$\begin{aligned} X_t &= Z_t^{0,\delta}, \quad & \varphi_s^t &= S_{s,t}R_sF(Z_{\lfloor s\rfloor_h}^{0,\delta}), \qquad & \Phi_s^t &= S_{s,t}R_sG(Z_{\lfloor s\rfloor_h}^{0,\delta}, \int_{\lfloor s\rfloor_h}^{s} B(Z_{\lfloor s\rfloor_h}^{0,\delta})\,\mathrm{d}W_u), \\ \bar{X}_t &= 0, & \bar{\varphi}_s^t &= S_{s,t}R_sF(0) \quad \text{and} & \bar{\Phi}_s^t &= S_{s,t}R_sG(0, \int_{\lfloor s\rfloor_h}^{s} B(0)\,\mathrm{d}W_u). \end{aligned}$$

By assumption, we have $\sup_{t\in[0,T]}\|X_t - \bar{X}_t\|_{L^r(\Omega,H)} < \infty$. Furthermore, we get that

$$\begin{aligned} \left\|\varphi_s^t - \bar{\varphi}_s^t\right\|_{L^r(\Omega,H)} &\leq C_\vartheta (t-s)^{-\vartheta}|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\|Z_{\lfloor s\rfloor_h}^{0,\delta}\|_{L^r(\Omega,H)} && (3.23)\\ &\leq C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{\sup_{u\in[0,s]}\|Z_u^{0,\delta}\|_{L^r(\Omega,H)}}{(t-s)^\vartheta} && (3.24) \end{aligned}$$

and using Assumption 2 also

$$\begin{aligned} \left\|\Phi_s^t - \bar{\Phi}_s^t\right\|_{L^r(\Omega,\mathrm{L}_2(U,H))} &\leq C_0C_G(1+\tau_r h^{\frac{1}{2}})\|Z_{\lfloor s\rfloor_h}^{0,\delta}\|_{L^r(\Omega,H)} && (3.25)\\ &\leq C_0C_G(1+\tau_r h^{\frac{1}{2}})T\frac{\sup_{u\in[0,s]}\|Z_u^{0,\delta}\|_{L^r(\Omega,H)}}{(t-s)^{\frac{\vartheta}{2}}}. && (3.26) \end{aligned}$$

Proposition 3.2 thus implies that

$$\begin{aligned} \sup_{t\in[0,T]}\|Z_t^{0,\delta}\|_{L^r(\Omega,H)} \leq \sqrt{2}\cdot \sup_{t\in[0,T]}\Big\| S_{0,t}Z_0^{0,\delta} &+ \int_0^t S_{s,t}R_sF(0)\,\mathrm{d}s \\ &+ \int_0^t S_{s,t}R_sG(0, \int_{\lfloor s\rfloor_h}^{s} B(0)\,\mathrm{d}W_u)\,\mathrm{d}W_s\Big\|_{L^r(\Omega,H)} \\ &\cdot \mathrm{E}_{1-\vartheta}\bigg(2\Gamma(2-\vartheta)\bigg(C_\vartheta|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\tfrac{T^{1-\vartheta}}{1-\vartheta} \\ &+ \tau_r C_0C_G(1+\tau_r h^{\frac{1}{2}})T\sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\bigg)^2\bigg)^{\frac{1}{2}}. \end{aligned} \tag{3.27}$$

Using the triangle inequality and further estimating the terms in the supremum as

$$\sup_{t\in[0,T]}\left\|S_{0,t}Z_0^{0,\delta}\right\|_{L^r(\Omega,H)} \leq C_0\|\xi_\delta\|_{L^r(\Omega,H)}, \tag{3.28}$$

$$\sup_{t\in[0,T]}\left\|\int_0^t S_{s,t}R_sF(0)\,\mathrm{d}s\right\|_{L^r(\Omega,H)} \leq \|F(0)\|_{H_{-\vartheta}}C_\vartheta\frac{T^{1-\vartheta}}{1-\vartheta} \tag{3.29}$$

and

$$\begin{aligned}
&\sup_{t\in[0,T]}\Big\| \int_0^t S_{s,t} R_s G(0, \int_{\lfloor s\rfloor_h}^s B(0)\,\mathrm{d}W_u)\,\mathrm{d}W_s \Big\|_{L^r(\Omega,H)} && (3.30)\\
&\quad\le \sup_{t\in[0,T]} \tau_r C_0 C_G \Big(\int_0^t \Big\| \max\Big\{1, \Big\| \int_{\lfloor s\rfloor_h}^s B(0)\,\mathrm{d}W_u \Big\|_H \Big\} \Big\|^2_{L^r(\Omega)}\,\mathrm{d}s \Big)^{\frac{1}{2}} && (3.31)\\
&\quad\le \sup_{t\in[0,T]} \tau_r C_0 C_G \Big(t^{\frac{1}{2}} + \tau_r \Big(\int_0^t \int_{\lfloor s\rfloor_h}^s \|B(0)\|^2_{\mathrm{L}_2(U,H)}\,\mathrm{d}u\,\mathrm{d}s \Big)^{\frac{1}{2}} \Big) && (3.32)\\
&\quad\le \sup_{t\in[0,T]} \tau_r C_0 C_G \Big(t^{\frac{1}{2}} + \tau_r \|B(0)\|_{\mathrm{L}_2(U,H)} h^{\frac{1}{2}} t^{\frac{1}{2}} \Big) && (3.33)\\
&\quad\le \tau_r C_0 C_G \Big(1 + \tau_r \|B(0)\|_{\mathrm{L}_2(U,H)} h^{\frac{1}{2}} \Big) T^{\frac{1}{2}} && (3.34)
\end{aligned}$$

completes the proof. □

Combining Lemma 3.4 with the last estimate in the proof of Lemma 2.2 gives the following explicit moment bound.

Corollary 3.5.
It holds that

$$\begin{aligned}
K_p \le 8\Big[& C_0 \max\{1, \|\xi\|_{L^p(\Omega,H)}\}\\
&+ C_\vartheta \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} + \tau_p C_0 C_G (1 + \tau_p \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) T^{\frac{1}{2}} \Big]^2\\
&\cdot \mathrm{E}_{1-\vartheta} \left(2\Gamma(2-\vartheta) \left(C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} + \tau_p C_0 C_G (1 + \tau_p T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}} \right)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

3.2.2 Mollification of the Coefficient Functions

We again apply Proposition 3.2 to obtain estimates for the errors resulting from the mollification of the coefficient functions. This is done separately for both X and Z. Here it is crucial for us that we get a good rate of convergence in the mollification parameter ε, as this will in the end affect the convergence rate of the whole scheme. If we could get an arbitrarily good rate, this would imply a convergence rate of $1/2$ for the whole scheme, independent of the regularity ϑ.

Lemma 3.6.
Let $r \ge 2$, $\varepsilon, \delta \in [0,T]$ *and* $\rho \in [0, \min\{1-\vartheta, \frac{1}{2}\})$. *Suppose that* $\sup_{t\in[0,T]} \|X_t^{0,\delta} - X_t^{\varepsilon,\delta}\|_{L^r(\Omega,H)} < \infty$. *Then we have*

$$
\begin{aligned}
\sup_{t\in[0,T]} &\|X_t^{0,\delta} - X_t^{\varepsilon,\delta}\|_{L^r(\Omega,H)} \\
&\leq 2\varepsilon^\rho T^{-\rho}\Big[C_0 \max\{1, \|\xi_\delta\|_{L^r(\Omega,H)}\} \\
&\qquad + C_\vartheta C_\rho C_{\vartheta+\rho}\|F\|_{\mathrm{Lip}(H,H_{-\vartheta})}\tfrac{T^{1-\vartheta}}{1-\vartheta-\rho} + \tau_r C_0 C_\rho^2 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\sqrt{\tfrac{T}{1-2\rho}}\Big]^2 \\
&\cdot \mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta)\left(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})}\tfrac{T^{1-\vartheta}}{1-\vartheta} + \tau_r C_0^2 |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T\sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\right)^2\right)
\end{aligned}
$$

Proof. We use Proposition 3.2 with

$$
\begin{aligned}
X_t &= X_t^{0,\delta}, & \varphi_s^t &= e^{(t-s)A}F_\varepsilon(X_s^{0,\delta}), & \Phi_s^t &= e^{(t-s)A}B_\varepsilon(X_s^{0,\delta}), \\
\bar{X}_t &= X_t^{\varepsilon,\delta}, & \bar{\varphi}_s^t &= e^{(t-s)A}F_\varepsilon(X_s^{\varepsilon,\delta}) \quad \text{and} & \bar{\Phi}_s^t &= e^{(t-s)A}B_\varepsilon(X_s^{\varepsilon,\delta}).
\end{aligned}
$$

By assumption, we have that $\sup_{t\in[0,T]}\|X_t - \bar{X}_t\|_{L^r(\Omega,H)} < \infty$. Furthermore, we get that

$$
\begin{aligned}
\left\|\varphi_s^t - \bar{\varphi}_s^t\right\|_{L^r(\Omega,H)} &\leq C_\vartheta (t-s)^{-\vartheta} C_0 |F|_{\mathrm{Lip}(H,H_{-\vartheta})}\|X_s^{0,\delta} - X_s^{\varepsilon,\delta}\|_{L^r(\Omega,H)} && (3.35) \\
&\leq C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{\sup_{u\in[0,s]}\|X_u^{0,\delta} - X_u^{\varepsilon,\delta}\|_{L^r(\Omega,H)}}{(t-s)^\vartheta} && (3.36)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\left\|\Phi_s^t - \bar{\Phi}_s^t\right\|_{L^r(\Omega,\mathrm{L}_2(U,H))} &\leq C_0^2 |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\|X_s^{0,\delta} - X_s^{\varepsilon,\delta}\|_{L^r(\Omega,H)} && (3.37) \\
&\leq C_0^2 |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T \frac{\sup_{u\in[0,s]}\|X_u^{0,\delta} - X_u^{\varepsilon,\delta}\|_{L^r(\Omega,H)}}{(t-s)^{\frac{\vartheta}{2}}}. && (3.38)
\end{aligned}
$$

Proposition 3.2 thus implies that

$$
\begin{aligned}
\sup_{t\in[0,T]}\|X_t^{0,\delta} - X_t^{\varepsilon,\delta}\|_{L^r(\Omega,H)} \leq \sqrt{2}\cdot &\sup_{t\in[0,T]}\Big\|\int_0^t e^{(t-s)A}(\mathrm{Id}_{H_{-\vartheta}} - e^{\varepsilon A})F(X_s^{0,\delta})\,\mathrm{d}s \\
&+ \int_0^t e^{(t-s)A}(\mathrm{Id}_H - e^{\varepsilon A})B(X_s^{0,\delta})\,\mathrm{d}W_s\Big\|_{L^r(\Omega,H)} \\
&\cdot \mathrm{E}_{1-\vartheta}\Bigg(2\Gamma(2-\vartheta)\Bigg(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{T^{1-\vartheta}}{1-\vartheta} \qquad (3.39) \\
&+ \tau_r C_0^2 |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T\sqrt{\frac{T^{1-\vartheta}}{1-\vartheta}}\Bigg)^2\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Next we use the triangle inequality and estimate

$$\begin{aligned}
&\left\| \int_0^t e^{(t-s)A} (\mathrm{Id}_{H_{-\vartheta}} - e^{\varepsilon A}) F(X_s^{0,\delta}) \, \mathrm{d}s \right\|_{L^r(\Omega,H)} && (3.40)\\
&\quad \le \int_0^t \| e^{(t-s)A} \|_{L(H_{-\vartheta-\rho},H)} \| \mathrm{Id}_{H_{-\vartheta}} - e^{\varepsilon A} \|_{L(H_{-\vartheta},H_{-\vartheta-\rho})} \| F(X_s^{0,\delta}) \|_{L^r(\Omega,H_{-\vartheta})} \, \mathrm{d}s && (3.41)\\
&\quad \le \int_0^t \frac{C_{\vartheta+\rho}}{(t-s)^{\vartheta+\rho}} \cdot C_\rho \varepsilon^\rho \cdot \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \max\{1, \|X_s^{0,\delta}\|_{L^r(\Omega,H)}\} \, \mathrm{d}s && (3.42)\\
&\quad \le C_{\vartheta+\rho} C_\rho \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \frac{t^{1-\vartheta-\rho}}{1-\vartheta-\rho} \max\{1, \sup_{s\in[0,T]} \|X_s^{0,\delta}\|_{L^r(\Omega,H)}\} \cdot \varepsilon^\rho && (3.43)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \int_0^t e^{(t-s)A} (\mathrm{Id}_H - e^{\varepsilon A}) B(X_s^{0,\delta}) \, \mathrm{d}W_s \right\|_{L^r(\Omega,H)} && (3.44)\\
&\quad \le \tau_r \left(\int_0^t \| e^{(t-s)A} \|^2_{L(H_{-\rho},H)} \| \mathrm{Id}_H - e^{\varepsilon A} \|^2_{L(H,H_{-\rho})} \| B(X_s^{0,\delta}) \|^2_{L^r(\Omega,\mathrm{L}_2(U,H))} \, \mathrm{d}s \right)^{\frac{1}{2}} && (3.45)\\
&\quad \le \tau_r \left(\int_0^t \frac{C_\rho^2}{(t-s)^{2\rho}} \cdot C_\rho^2 \varepsilon^{2\rho} \cdot \|B\|^2_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} \max\{1, \|X_s^{0,\delta}\|_{L^r(\Omega,H)}\}^2 \, \mathrm{d}s \right)^{\frac{1}{2}} && (3.46)\\
&\quad \le \tau_r C_\rho^2 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} \frac{t^{\frac{1}{2}-\rho}}{\sqrt{1-2\rho}} \max\{1, \sup_{s\in[0,T]} \|X_s^{0,\delta}\|_{L^r(\Omega,H)}\} \cdot \varepsilon^\rho && (3.47)
\end{aligned}$$

which requires $\rho < 1-\vartheta$ and $\rho < \frac{1}{2}$, respectively. Combining this with Lemma 3.3 we arrive at

$$\begin{aligned}
&\sup_{t\in[0,T]} \|X_t^{0,\delta} - X_t^{\varepsilon,\delta}\|_{L^r(\Omega,H)}\\
&\quad \le 2\varepsilon^\rho T^{-\rho} \Big[C_0 \max\{1, \|\xi_\delta\|_{L^r(\Omega,H)}\}\\
&\quad + C_\vartheta C_\rho C_{\vartheta+\rho} \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta-\rho} + \tau_r C_0 C_\rho^2 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} \sqrt{\tfrac{T}{1-2\rho}} \Big]^2\\
&\quad \cdot \mathrm{E}_{1-\vartheta} \left(2\Gamma(2-\vartheta) \left(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} + \tau_r C_0^2 |B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}} \right)^2 \right)
\end{aligned}$$
(3.48)

□

Lemma 3.7.
Let $r \ge 2$, $\varepsilon, \delta \in [0,T]$ and $\rho \in [0, \min\{1-\vartheta, \frac{1}{2}\})$. Suppose that $\sup_{t\in[0,T]} \|Z_t^{0,\delta} - Z_t^{\varepsilon,\delta}\|_{L^r(\Omega,H)} < \infty$. Then we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \|Z_t^{0,\delta} - Z_t^{\varepsilon,\delta}\|_{L^r(\Omega,H)} \\
&\quad\le 2\varepsilon^{\rho} T^{-\rho} \Big[C_0 \max\{1, \|\xi_\delta\|_{L^r(\Omega,H)}\} + C_\vartheta C_\rho C_{\vartheta+\rho} \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta-\rho} \\
&\quad + 2\tau_r C_0 C_\rho^2 C_G (1 + \tau_r \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} h^{\frac{1}{2}}) \sqrt{\tfrac{T}{1-2\rho}} \Big]^2 \\
&\quad \cdot \mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta) \left(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} + \tau_r C_0^2 C_G (1 + \tau_r h^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}} \right)^2 \right)
\end{aligned}$$

Proof. We use Proposition 3.2 with

$$\begin{aligned}
X_t &= Z_t^{0,\delta}, \quad \varphi_s^t = S_{s,t} R_s F_\varepsilon(Z_{\lfloor s\rfloor_h}^{0,\delta}), \qquad \Phi_s^t = S_{s,t} R_s G_\varepsilon\Big(Z_{\lfloor s\rfloor_h}^{0,\delta}, \int_{\lfloor s\rfloor_h}^s B(Z_{\lfloor s\rfloor_h}^{0,\delta})\,\mathrm{d}W_u\Big), \\
\bar{X}_t &= Z_t^{\varepsilon,\delta}, \quad \bar{\varphi}_s^t = S_{s,t} R_s F_\varepsilon(Z_{\lfloor s\rfloor_h}^{\varepsilon,\delta}) \quad \text{and} \quad \bar{\Phi}_s^t = S_{s,t} R_s G_\varepsilon\Big(Z_{\lfloor s\rfloor_h}^{\varepsilon,\delta}, \int_{\lfloor s\rfloor_h}^s B(Z_{\lfloor s\rfloor_h}^{\varepsilon,\delta})\,\mathrm{d}W_u\Big).
\end{aligned}$$

By assumption, we have that $\sup_{t\in[0,T]} \|X_t - \bar{X}_t\|_{L^r(\Omega,H)} < \infty$. Further we get that

$$\begin{aligned}
\Big\|\varphi_s^t - \bar{\varphi}_s^t\Big\|_{L^r(\Omega,H)} &\le C_\vartheta (t-s)^{-\vartheta} C_0 |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \|Z_{\lfloor s\rfloor_h}^{0,\delta} - Z_{\lfloor s\rfloor_h}^{\varepsilon,\delta}\|_{L^r(\Omega,H)} && (3.49)\\
&\le C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \frac{\sup_{u\in[0,s]} \|Z_u^{0,\delta} - Z_u^{\varepsilon,\delta}\|_{L^r(\Omega,H)}}{(t-s)^\vartheta} && (3.50)
\end{aligned}$$

as well as

$$\begin{aligned}
\Big\|\Phi_s^t - \bar{\Phi}_s^t\Big\|_{L^r(\Omega,\mathrm{L}_2(U,H))} &\le C_0^2 C_G (1 + \tau_r h^{\frac{1}{2}}) \|Z_{\lfloor s\rfloor_h}^{0,\delta} - Z_{\lfloor s\rfloor_h}^{\varepsilon,\delta}\|_{L^r(\Omega,H)} && (3.51)\\
&\le C_0^2 C_G (1 + \tau_r h^{\frac{1}{2}}) T \frac{\sup_{u\in[0,s]} \|Z_u^{0,\delta} - Z_u^{\varepsilon,\delta}\|_{L^r(\Omega,H)}}{(t-s)^{\frac{\vartheta}{2}}}. && (3.52)
\end{aligned}$$

Proposition 3.2 thus implies that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|Z_t^{0,\delta} - Z_t^{\varepsilon,\delta}\|_{L^r(\Omega,H)} \\
&\quad\le \sqrt{2} \cdot \sup_{t\in[0,T]} \Big\| \int_0^t S_{s,t} R_s (\mathrm{Id}_{H_{-\vartheta}} - e^{\varepsilon A}) F(Z_{\lfloor s\rfloor_h}^{0,\delta})\,\mathrm{d}s \\
&\qquad + \int_0^t S_{s,t} R_s (\mathrm{Id}_H - e^{\varepsilon A}) G\Big(Z_{\lfloor s\rfloor_h}^{0,\delta}, \int_{\lfloor s\rfloor_h}^s B(Z_{\lfloor s\rfloor_h}^{0,\delta})\,\mathrm{d}W_u\Big)\,\mathrm{d}W_s \Big\|_{L^r(\Omega,H)} && (3.53)\\
&\qquad \cdot \mathrm{E}_{1-\vartheta}\Bigg(2\Gamma(2-\vartheta) \Bigg(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \frac{T^{1-\vartheta}}{1-\vartheta} \\
&\qquad + \tau_r C_0^2 C_G (1 + \tau_r h^{\frac{1}{2}}) T \sqrt{\frac{T^{1-\vartheta}}{1-\vartheta}} \Bigg)^2 \Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Next we use the triangle inequality and estimate

$$
\begin{aligned}
&\left\|\int_0^t S_{s,t}R_s(\mathrm{Id}_{H_{-\vartheta}} - e^{\varepsilon A})F(Z^{0,\delta}_{\lfloor h\rfloor_h})\,\mathrm{d}s\right\|_{L^r(\Omega,H)} \\
&\quad\leq \int_0^t \|S_{s,t}R_s\|_{L(H_{-\vartheta-\rho},H)}\|\mathrm{Id}_{H_{-\vartheta}} - e^{\varepsilon A}\|_{L(H_{-\vartheta},H_{-\vartheta-\rho})}\|F(Z^{0,\delta}_{\lfloor s\rfloor_h})\|_{L^r(\Omega,H_{-\vartheta})}\,\mathrm{d}s \qquad (3.54)\\
&\quad\leq \int_0^t \frac{C_{\vartheta+\rho}}{(t-s)^{\vartheta+\rho}} \cdot C_\rho\varepsilon^\rho \cdot \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})}\max\{1,\|Z^{0,\delta}_{\lfloor s\rfloor_h}\|_{L^r(\Omega,H)}\}\,\mathrm{d}s \qquad (3.55)\\
&\quad\leq C_{\vartheta+\rho}C_\rho\|F\|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{t^{1-\vartheta-\rho}}{1-\vartheta-\rho}\max\{1,\sup_{s\in[0,T]}\|Z^{0,\delta}_{\lfloor s\rfloor_h}\|_{L^r(\Omega,H)}\}\cdot\varepsilon^\rho \qquad (3.56)
\end{aligned}
$$

and using Lemma 2.1 ii)

$$
\begin{aligned}
&\left\|\int_0^t S_{s,t}R_s(\mathrm{Id}_H - e^{\varepsilon A})G\left(Z^{0,\delta}_{\lfloor s\rfloor_h},\int_{\lfloor s\rfloor_h}^s B(Z^{0,\delta}_{\lfloor s\rfloor_h})\,\mathrm{d}W_u\right)\mathrm{d}W_s\right\|_{L^r(\Omega,H)} \\
&\quad\leq \tau_r\Bigg(\int_0^t \|S_{s,t}R_s\|^2_{L(H_{-\rho},H)}\|\mathrm{Id}_H - e^{\varepsilon A}\|^2_{L(H,H_{-\rho})} \\
&\qquad\cdot\|G(Z^{0,\delta}_{\lfloor s\rfloor_h},\int_{\lfloor s\rfloor_h}^s B(Z^{0,\delta}_{\lfloor s\rfloor_h})\,\mathrm{d}W_u)\|^2_{L^r(\Omega,\mathrm{L}_2(U,H))}\,\mathrm{d}s\Bigg)^{\frac{1}{2}} \qquad (3.57)\\
&\quad\leq \tau_r\Bigg(\int_0^t \frac{C_\rho^2}{(t-s)^{2\rho}}\cdot C_\rho^2\varepsilon^{2\rho}\cdot C_G^2(1+\tau_r\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}h^{\frac{1}{2}})^2 \\
&\qquad\cdot\|\max\{1,\|Z^{0,\delta}_{\lfloor s\rfloor_h}\|_H\}\|^2_{L^r(\Omega)}\,\mathrm{d}s\Bigg)^{\frac{1}{2}} \qquad (3.58)\\
&\quad\leq 2\tau_r C_\rho^2 C_G(1+\tau_r\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}h^{\frac{1}{2}})\frac{t^{\frac{1}{2}-\rho}}{\sqrt{1-2\rho}}\max\{1,\sup_{s\in[0,T]}\|Z^{0,\delta}_{\lfloor s\rfloor_h}\|_{L^r(\Omega,H)}\}\cdot\varepsilon^\rho \qquad (3.59)
\end{aligned}
$$

which requires $\rho < 1-\vartheta$ and $\rho < \frac{1}{2}$, respectively. Combining this with Lemma 3.4 we arrive at

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|Z^{0,\delta}_t - Z^{\varepsilon,\delta}_t\|_{L^r(\Omega,H)} \\
&\quad\leq 2\varepsilon^\rho T^{-\rho}\Bigg[C_0\max\{1,\|\xi_\delta\|_{L^r(\Omega,H)}\} + C_\vartheta C_\rho C_{\vartheta+\rho}\|F\|_{\mathrm{Lip}(H,H_{-\vartheta})}\tfrac{T^{1-\vartheta}}{1-\vartheta-\rho} \\
&\quad+ 2\tau_r C_0 C_\rho^2 C_G(1+\tau_r\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}h^{\frac{1}{2}})\sqrt{\tfrac{T}{1-2\rho}}\Bigg]^2 \\
&\quad\cdot \mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta)\left(C_0C_\vartheta|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\tfrac{T^{1-\vartheta}}{1-\vartheta} + \tau_r C_0^2 C_G(1+\tau_r h^{\frac{1}{2}})T\sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\right)^2\right) \qquad (3.60)
\end{aligned}
$$

□

3.2.3 Mollification of the Initial Value

We use Proposition 3.2 once more, to also estimate the errors from the mollification of the initial value. However, this time we are not interested in the rates of convergence. It will be sufficient for us to know that these errors vanish as the mollification parameter δ goes to zero. Again, we treat X and Z separately.

Lemma 3.8.
Let $r \in [2,p]$*. Suppose that* $\sup_{t\in[0,T]}\|X_t - X_t^{0,\delta}\|_{L^r(\Omega,H)} < \infty$*. Then we have*

$$\lim_{\delta\to 0}\sup_{t\in[0,T]}\|X_t - X_t^{0,\delta}\|_{L^r(\Omega,H)} = 0.$$

Proof. Let us first fix an arbitrary $\delta \in [0,T]$. We use Proposition 3.2 with

$$\begin{aligned} X_t &= X_t, & \varphi_s^t &= e^{(t-s)A}F(X_s), & \Phi_s^t &= e^{(t-s)A}B(X_s),\\ \bar{X}_t &= X_t^{0,\delta}, & \bar{\varphi}_s^t &= e^{(t-s)A}F(X_s^{0,\delta}) \quad\text{and} & \bar{\Phi}_s^t &= e^{(t-s)A}B(X_s^{0,\delta}). \end{aligned}$$

By assumption, we have $\sup_{t\in[0,T]}\|X_t - \bar{X}_t\|_{L^r(\Omega,H)} < \infty$. Further we get that

$$\begin{aligned} \left\|\varphi_s^t - \bar{\varphi}_s^t\right\|_{L^r(\Omega,H)} &\le C_\vartheta (t-s)^{-\vartheta}|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\|X_s - X_s^{0,\delta}\|_{L^r(\Omega,H)} && (3.61)\\ &\le C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{\sup_{u\in[0,s]}\|X_u - X_u^{0,\delta}\|_{L^r(\Omega,H)}}{(t-s)^{\vartheta}} && (3.62) \end{aligned}$$

as well as

$$\begin{aligned} \left\|\Phi_s^t - \bar{\Phi}_s^t\right\|_{L^r(\Omega,\mathrm{L}_2(U,H))} &\le C_0|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\|X_s - X_s^{0,\delta}\|_{L^r(\Omega,H)} && (3.63)\\ &\le C_0|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}T\frac{\sup_{u\in[0,s]}\|X_u - X_u^{0,\delta}\|_{L^r(\Omega,H)}}{(t-s)^{\frac{\vartheta}{2}}}. && (3.64) \end{aligned}$$

Proposition 3.2 thus implies that

$$\begin{aligned} &\sup_{t\in[0,T]}\|X_t - X_t^{0,\delta}\|_{L^r(\Omega,H)}\\ &\quad\le \sqrt{2}\cdot\sup_{t\in[0,T]}\left\|e^{tA}\left(\xi - \xi_\delta\right)\right\|_{L^r(\Omega,H)}\\ &\quad\cdot \mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta)\left(C_\vartheta|F|_{\mathrm{Lip}(H,H_{-\vartheta})}\frac{T^{1-\vartheta}}{1-\vartheta} + \tau_r C_0|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}T\sqrt{\frac{T^{1-\vartheta}}{1-\vartheta}}\right)^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.65}$$

Furthermore, we have $\sup_{t\in[0,T]}\left\|e^{tA}\left(\xi - \xi_\delta\right)\right\|_{L^r(\Omega,H)} \le C_0\|\xi - \xi_\delta\|_{L^r(\Omega,H)}$.

Letting δ go to zero we see that almost surely

$$\lim_{\delta\to 0}\|\xi - \xi_\delta\|_H = \lim_{\delta\to 0}\|\xi - e^{\delta A}\xi\|_H = 0 \tag{3.66}$$

by the properties of the analytic semigroup (Definition 1.16 iii)). Additionally, we have almost surely for all $\delta \in [0,T]$

$$\|\xi - \xi_\delta\|_H = \|(\mathrm{Id}_H - e^{\delta A})\xi\|_H \leq C_0\|\xi\|_H \tag{3.67}$$

as well as $\|\xi\|_{L^r(\Omega,H)} \leq \|\xi\|_{L^p(\Omega,H)} < \infty$. Thus, we can apply the dominated convergence theorem [Coh13, Thm. E.6] to conclude that

$$\lim_{\delta\to 0} \|\xi - \xi_\delta\|_{L^r(\Omega,H)} = 0 \tag{3.68}$$

□

Lemma 3.9.
Let $r \in [2,p]$. Suppose that $\sup_{t\in[0,T]}\|Z_t - Z_t^{0,\delta}\|_{L^r(\Omega,H)} < \infty$. Then we have

$$\lim_{\delta\to 0}\sup_{t\in[0,T]}\|Z_t - Z_t^{0,\delta}\|_{L^r(\Omega,H)} = 0.$$

Proof. The proof is analogous to the proof of Lemma 3.8 but with

$$\Phi_s^t = S_{s,t}R_sG(Z_{\lfloor s\rfloor_h}, \int_{\lfloor s\rfloor_h}^s B(Z_{\lfloor s\rfloor_h})\,\mathrm{d}W_u)$$

and

$$\bar{\Phi}_s^t = S_{s,t}R_sG(Z^{0,\delta}_{\lfloor s\rfloor_h}, \int_{\lfloor s\rfloor_h}^s B(Z^{0,\delta}_{\lfloor s\rfloor_h})\,\mathrm{d}W_u)$$

leading to

$$\left\|\Phi_s^t - \bar{\Phi}_s^t\right\|_{L^r(\Omega,\mathrm{L}_2(U,H))} \leq C_0C_G(1+\tau_r h^{\frac{1}{2}})T\frac{\sup_{u\in[0,s]}\|X_u - X_u^{0,\delta}\|_{L^r(\Omega,H)}}{(t-s)^{\frac{\vartheta}{2}}}. \tag{3.69}$$

The rest of the proof is, mutatis mutandis, the same. □

3.3 Weak Convergence for Irregular Coefficients

After the preparations in section 3.2, we will now perform the necessary mollification of the coefficients of the SEE as well as of the initial value. We have proved above, that the mollified processes are still close to the original processes and thus we can substitute them in our error criterion. Since the mollified processes satisfy the assumptions of chapter 2, we can then apply Corollary 2.10 to prove the weak convergence. Unfortunately, it holds only in the case of $\vartheta \in [0,1/2)$.

The restrictions on the range of ϑ in Corollary 2.10 originate from Lemma 2.8 and ultimately from Theorem 1.37 and are an active research topic (see e.g. [And+19; BD18; Deb11]). Working with the result at hand, we can achieve the optimal order of convergence of $1-\vartheta$ only in the restricted range $\vartheta \in [0,1/2)$. For $\vartheta \in [1/2,1)$ the used mollification strategy causes a reduction in the order relative to the expected optimal order of up to two thirds in the limiting case $\vartheta \to 1$. Extending Theorem 1.37 to include $\delta_1,\ldots,\delta_k \in [0,1)$ and $\sum_{i=1}^k \delta_i < 1$ would unify the two cases below and yield the optimal order $1-\vartheta$ for all $\vartheta \in [0,1)$.

3.3.1 The Case $\vartheta \in [0, 1/2)$

In this case the coefficients are still regular enough to achieve the optimal order. Nevertheless, we need some kind of regularisation to extend Corollary 2.10 to the full generality presented here.

Theorem 3.10.
Let $\vartheta \in [0, \frac{1}{2})$, $\rho \in [0, 1-\vartheta)$, $p \geq 5$ and $\varphi \in \mathrm{Lip}^4(H, V)$. Then there exists a constant $C \in [0, \infty)$ such that

$$\|\mathbb{E}[\varphi(Z_T) - \varphi(X_T)]\|_V \leq C \cdot h^\rho.$$

Proof. We proof the result first for mollified initial values, i.e. $\delta > 0$, and in the end use the dominated convergence theorem to pass to the limit $\delta \to 0$. To this end, we insert the mollified ($\varepsilon > 0$) processes and estimate separately the distance between the mollified and non-mollified processes by Lemmas 3.6 and 3.7 and the distance between the two mollified processes by the results from chapter 2, in particular Corollary 2.10.

The moral here is, that we can get the convergence of the mollified processes to be independent of the mollification parameter ε and thus we can choose it sufficiently small to not destroy the overall convergence.

We use the mollified processes defined above and the triangle inequality to get that

$$\begin{aligned}\Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \leq{}& \Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(Z_T^{\varepsilon,\delta})\Big]\Big\|_V \\ &+ \Big\|\mathbb{E}\Big[\varphi(Z_T^{\varepsilon,\delta}) - \varphi(X_T^{\varepsilon,\delta})\Big]\Big\|_V \\ &+ \Big\|\mathbb{E}\Big[\varphi(X_T^{\varepsilon,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V\end{aligned} \tag{3.70}$$

for all $\varepsilon, \delta \in [0, T]$. Since $\sup_{t\in[0,T]} \|X_t^{\varepsilon,\delta}\|_{L^p(\Omega,H)} < \infty$ as well as $\sup_{t\in[0,T]} \|Z_t^{\varepsilon,\delta}\|_{L^p(\Omega,H)} < \infty$ for all $\varepsilon, \delta \in [0, T]$ we can apply Lemmas 3.6 and 3.7 with $r = 2$ and $\rho = \rho/2$ to estimate

$$\begin{aligned}&\Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(Z_T^{\varepsilon,\delta})\Big]\Big\|_V + \Big\|\mathbb{E}\Big[\varphi(X_T^{\varepsilon,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \\ &\quad\leq |\varphi|_{\mathrm{Lip}(H,V)} \cdot \left(\sup_{t\in[0,T]} \|Z_t^{0,\delta} - Z_t^{\varepsilon,\delta}\|_{L^2(\Omega,H)} + \sup_{t\in[0,T]} \|X_t^{0,\delta} - X_t^{\varepsilon,\delta}\|_{L^2(\Omega,H)}\right) \\ &\quad\leq 4|\varphi|_{\mathrm{Lip}(H,V)} \varepsilon^{\rho/2} T^{-\rho/2} \Big[C_0 \max\{1, \|\xi_\delta\|_{L^2(\Omega,H)}\} \\ &\qquad + C_\vartheta C_{\rho/2} C_{\vartheta+\rho/2} \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta-\rho/2} \\ &\qquad + 2\tau_2 C_0 C_{\rho/2}^2 \max\{\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_2 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) \sqrt{\tfrac{T}{1-\rho}}\Big]^2 \\ &\qquad \cdot \mathrm{E}_{1-\vartheta}\Big(2\Gamma(2-\vartheta)\Big(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} \\ &\qquad + \tau_2 C_0^2 \max\{|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_2 T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\Big)^2\Big)\end{aligned} \tag{3.71}$$

for all $\varepsilon, \delta \in [0, T]$. Next we use Corollary 2.10 with $\rho = \rho$ and $\vartheta = \vartheta$ to bound the second term for all $\varepsilon, \delta \in (0, T]$ as

$$
\begin{aligned}
&\Big\|\mathbb{E}\Big[\varphi(Z_T^{\varepsilon,\delta}) - \varphi(X_T^{\varepsilon,\delta})\Big]\Big\|_V \\
&\quad\leq h^\rho \cdot C_{G_\varepsilon}^7 \cdot \max\{1, \|F_\varepsilon\|_{C_b^3(H,H_{-\vartheta})}\}^3 \cdot \max\{1, \|B_\varepsilon\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\}^7 \\
&\qquad \cdot 288\Big[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + c_\vartheta^{(\varepsilon)} + c_{\vartheta,0}^{(\varepsilon)} + c_{\vartheta,0,0}^{(\varepsilon)} + c_{\vartheta,0,0,0}^{(\varepsilon)} \\
&\qquad + c_{0,0}^{(\varepsilon)} + c_{0,0,0}^{(\varepsilon)} + c_{0,0,0,0}^{(\varepsilon)} + \tilde{c}_{0,0,0,0}^{(\varepsilon)}\Big] \\
&\qquad \cdot C_0^4\Big[C_\rho^2 C_{\rho+\vartheta} C_\vartheta + C_{0,0,\rho} + C_{\vartheta,0,\rho} + C_\vartheta C_{0,\rho} \\
&\qquad + C_\vartheta C_\rho (C_{\rho,\rho} + C_{\vartheta,-\rho,\rho} + \tau_5 C_{0,-\rho,\rho})\Big] \\
&\qquad \cdot \max\left\{1, \frac{1}{T}\right\} \cdot \left(1 + \tau_5 T^{1/2}\right)^7 \cdot \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right)^3 \cdot K_5^5
\end{aligned}
\tag{3.72}
$$

where Corollary 3.5 shows that

$$
\begin{aligned}
K_5 \leq 8\Big[&C_0 \max\{1, \|\xi_\delta\|_{L^5(\Omega,H)}\} + C_\vartheta \|F_\varepsilon\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} \\
&+ \tau_5 C_0 C_{G_\varepsilon} (1 + \tau_5 \|B_\varepsilon\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) T^{\frac{1}{2}}\Big]^2 \\
&\cdot \mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta)\left(C_\vartheta |F_\varepsilon|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} + \tau_5 C_0 C_{G_\varepsilon} (1 + \tau_5 T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.73}
$$

Because F is genuinely $H_{-\vartheta}$-valued we can bound $\|F_\varepsilon\|_{C_b^3(H,H_{-\vartheta})} \leq C_0 \|F\|_{C_b^3(H,H_{-\vartheta})}$ for all $\varepsilon \in [0, T]$ independent of ε and similarly for B.

Putting (3.70)–(3.73) together we arrive at

$$
\begin{aligned}
&\Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \\
&\quad\leq \left(\left(\frac{\varepsilon}{T}\right)^{\frac{\rho}{2}} + h^\rho\right) \cdot 2304 \cdot C_0^{51} \cdot C_G^7 \cdot \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^3 \\
&\qquad \cdot \max\{1, \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\}^7 \\
&\qquad \cdot \max\left\{1, \frac{1}{T}\right\} \cdot \Big(1 + \tau_5 T^{1/2}\Big)^7 \cdot \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right)^3 \\
&\qquad \cdot \Big[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + c_\vartheta^{(\varepsilon)} + c_{\vartheta,0}^{(\varepsilon)} + c_{\vartheta,0,0}^{(\varepsilon)} + c_{\vartheta,0,0,0}^{(\varepsilon)} + c_{0,0}^{(\varepsilon)} + c_{0,0,0}^{(\varepsilon)} + c_{0,0,0,0}^{(\varepsilon)} + \tilde{c}_{0,0,0,0}^{(\varepsilon)}\Big] \\
&\qquad \cdot \Big[C_\rho^2 C_{\rho+\vartheta} C_\vartheta + C_{0,0,\rho} + C_{\vartheta,0,\rho} + C_\vartheta C_{0,\rho} + C_\vartheta C_\rho (C_{\rho,\rho} + C_{\vartheta,-\rho,\rho} + \tau_5 C_{0,-\rho,\rho})\Big] \\
&\qquad \cdot \Bigg[\max\{1, \|\xi\|_{L^5(\Omega,H)}\} + C_\vartheta C_{\rho/2} C_{\vartheta+\rho/2} \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta-\rho/2} + 2\tau_5 C_{\rho/2}^2 \\
&\qquad \cdot \max\{\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_5 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) \sqrt{\tfrac{T}{1-\rho}}\Bigg]^{10} \\
&\qquad \cdot \mathrm{E}_{1-\vartheta}\Bigg(2\Gamma(2-\vartheta)\Big(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} \\
&\qquad + \tau_5 C_0^2 \max\{|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_5 T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\Big)^2\Bigg)^{\frac{5}{2}}
\end{aligned}
\tag{3.74}
$$

which holds for all $\varepsilon, \delta \in (0,T]$. If we now couple the mollification to the step size of the approximation as $\varepsilon = h^2 T$ and simultaneously use Theorem 1.38 to see that the constants arising from the Kolmogorov backward equation are bounded for $\varepsilon \in (0,T]$ then we obtain for all $\delta \in (0,T]$

$$\begin{aligned}
&\Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \\
&\quad \leq h^\rho \cdot 4608 \cdot C_0^{51} \cdot C_G^7 \cdot \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^3 \cdot \max\{1, \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\}^7 \\
&\qquad \cdot \max\left\{1, \frac{1}{T}\right\} \cdot \Big(1 + \tau_5 T^{1/2}\Big)^7 \cdot \left(1 + \frac{T^{1-\vartheta}}{1-\vartheta-\rho} + \frac{T}{1-\rho}\right)^3 \\
&\qquad \cdot \Bigg[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + \sup_{\varepsilon\in(0,T]} \Big[c_\vartheta^{(\varepsilon)} + c_{\vartheta,0}^{(\varepsilon)} + c_{\vartheta,0,0}^{(\varepsilon)} + c_{\vartheta,0,0,0}^{(\varepsilon)} \\
&\qquad + c_{0,0}^{(\varepsilon)} + c_{0,0,0}^{(\varepsilon)} + c_{0,0,0,0}^{(\varepsilon)} + \tilde{c}_{0,0,0,0}^{(\varepsilon)}\Big]\Bigg] \\
&\qquad \cdot \Big[C_\rho^2 C_{\rho+\vartheta} C_\vartheta + C_{0,0,\rho} + C_{\vartheta,0,\rho} + C_\vartheta C_{0,\rho} + C_\vartheta C_\rho (C_{\rho,\rho} + C_{\vartheta,-\rho,\rho} + \tau_5 C_{0,-\rho,\rho})\Big] \\
&\qquad \cdot \Bigg[\max\{1, \|\xi\|_{L^5(\Omega,H)}\} + C_\vartheta C_{\rho/2} C_{\vartheta+\rho/2} \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta-\rho/2} \\
&\qquad + 2\tau_5 C_{\rho/2}^2 \max\{\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_5 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) \sqrt{\tfrac{T}{1-\rho}}\Bigg]^{10} \\
&\qquad \cdot \mathrm{E}_{1-\vartheta}\Bigg(2\Gamma(2-\vartheta)\Bigg(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} \\
&\qquad + \tau_5 C_0^2 \max\{|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_5 T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\Bigg)^2\Bigg)^{\frac{5}{2}}.
\end{aligned} \tag{3.75}$$

Because the right-hand side of (3.75) does not depend on δ and since

$$\begin{aligned}
&\Big\|\mathbb{E}\Big[\varphi(Z_T) - \varphi(Z_T^{0,\delta})\Big]\Big\|_V + \Big\|\mathbb{E}\Big[\varphi(X_T^{0,\delta}) - \varphi(X_T)\Big]\Big\|_V \\
&\quad \leq |\varphi|_{\mathrm{Lip}(H,V)} \Big\|Z_T - Z_T^{0,\delta}\Big\|_{L^2(\Omega,H)} + |\varphi|_{\mathrm{Lip}(H,V)} \Big\|X_T^{0,\delta} - X_T\Big\|_{L^2(\Omega,H)}
\end{aligned} \tag{3.76}$$

we can use Lemmas 3.8 and 3.9 together with (3.75) to get that

$$\begin{aligned}
\|\mathbb{E}[\varphi(Z_T) - \varphi(X_T)]\|_V &\leq \lim_{\delta\to 0} \Big\|\mathbb{E}\Big[\varphi(Z_T) - \varphi(Z_T^{0,\delta})\Big]\Big\|_V \\
&\quad + \Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \qquad (3.77) \\
&\quad + \Big\|\mathbb{E}\Big[\varphi(X_T^{0,\delta}) - \varphi(X_T)\Big]\Big\|_V \\
&\leq C \cdot h^\rho \qquad (3.78)
\end{aligned}$$

where $C \in [0, \infty)$ is the constant on the right-hand side of (3.75). □

3.3.2 The Case $\vartheta \in [1/2, 1)$

In this irregular case, we find that the kind of regularisation we employed in the previous case would lead to a blow-up. We cannot remove the regularisation as fast as we would like and thus have to accept a reduction in the order of convergence. But, we can at least try to balance the step size and the mollification parameter in an optimal way. Thus, we have the following result.

Theorem 3.11.
Let $\vartheta \in [\frac{1}{2}, 1)$, $\rho \in [0, \frac{1-\vartheta}{4\vartheta-1})$, $p \geq 5$ *and* $\varphi \in \operatorname{Lip}^4(H, V)$. *Then there exists a constant* $C \in [0, \infty)$ *such that*

$$\|\mathbb{E}[\varphi(Z_T) - \varphi(X_T)]\|_V \leq C \cdot h^\rho.$$

Proof. The strategy is the same as for Theorem 3.10, but we have to choose some parameters differently since Lemma 2.8 only holds for $\vartheta \in [0, \frac{1}{2})$.

First, we use the mollified processes defined above and the triangle inequality to get that

$$\begin{aligned}\Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \leq{}& \Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(Z_T^{\varepsilon,\delta})\Big]\Big\|_V \\ &+ \Big\|\mathbb{E}\Big[\varphi(Z_T^{\varepsilon,\delta}) - \varphi(X_T^{\varepsilon,\delta})\Big]\Big\|_V \\ &+ \Big\|\mathbb{E}\Big[\varphi(X_T^{\varepsilon,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V\end{aligned} \tag{3.79}$$

for all $\varepsilon, \delta \in [0, T]$. Since $\sup_{t\in[0,T]} \|X_t^{\varepsilon,\delta}\|_{L^p(\Omega,H)} < \infty$ as well as $\sup_{t\in[0,T]} \|Z_t^{\varepsilon,\delta}\|_{L^p(\Omega,H)} < \infty$ for all $\varepsilon, \delta \in [0, T]$ we can apply Lemmas 3.6 and 3.7. In this case we want to apply them with $r = 2$ and $\rho = 1 - \vartheta - \epsilon$ where we fix $\epsilon = 1 - \vartheta - (4\vartheta - 1)\rho \in (0, 1 - \vartheta]$. The choice of ϵ will become clear later. We thus obtain

$$\begin{aligned}&\Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(Z_T^{\varepsilon,\delta})\Big]\Big\|_V + \Big\|\mathbb{E}\Big[\varphi(X_T^{\varepsilon,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \\ &\quad\leq |\varphi|_{\operatorname{Lip}(H,V)} \cdot \left(\sup_{t\in[0,T]} \|Z_t^{0,\delta} - Z_t^{\varepsilon,\delta}\|_{L^2(\Omega,H)} + \sup_{t\in[0,T]} \|X_t^{0,\delta} - X_t^{\varepsilon,\delta}\|_{L^2(\Omega,H)}\right) \\ &\quad\leq 4|\varphi|_{\operatorname{Lip}(H,V)} \varepsilon^{1-\vartheta-\epsilon} \max\left\{1, \frac{1}{T}\right\} \Big[C_0 \max\{1, \|\xi_\delta\|_{L^2(\Omega,H)}\} \\ &\qquad + C_\vartheta C_{1-\vartheta-\epsilon} C_{1-\epsilon} \|F\|_{\operatorname{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{\epsilon} + 2\tau_2 C_0 C_{1-\vartheta-\epsilon}^2 \\ &\qquad \cdot \max\{\|B\|_{\operatorname{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_2 \|B\|_{\operatorname{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) \sqrt{\tfrac{T}{2\vartheta-1+2\epsilon}}\Big]^2 \\ &\qquad \cdot \mathrm{E}_{1-\vartheta}\Big(2\Gamma(2-\vartheta)\Big(C_0 C_\vartheta |F|_{\operatorname{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} \\ &\qquad + \tau_2 C_0^2 \max\{|B|_{\operatorname{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_2 T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\Big)^2\Big)\end{aligned} \tag{3.80}$$

for all $\varepsilon, \delta \in [0, T]$. Next we use Corollary 2.10 with $\rho = \frac{1}{2}$ and $\vartheta = \frac{1}{2} - \frac{1}{3}\epsilon$ to bound the second term for all $\varepsilon, \delta \in (0, T]$ as

$$
\begin{aligned}
&\Big\|\mathbb{E}\Big[\varphi(Z_T^{\varepsilon,\delta}) - \varphi(X_T^{\varepsilon,\delta})\Big]\Big\|_V \\
&\quad\le h^{\frac{1}{2}} \cdot C_{G_\varepsilon}^7 \max\{1, \|F_\varepsilon\|_{C_b^3(H,H_{-\frac{1}{2}+\frac{1}{3}\epsilon})}\}^3 \cdot \max\{1, \|B_\varepsilon\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\}^7 \\
&\qquad \cdot 288\Big[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0,0} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0,0,0} \\
&\qquad + c^{(\varepsilon)}_{0,0} + c^{(\varepsilon)}_{0,0,0} + c^{(\varepsilon)}_{0,0,0,0} + \tilde{c}^{(\varepsilon)}_{0,0,0,0}\Big] \\
&\qquad \cdot C_0^4\Big[C_{\frac{1}{2}}^2 C_{1-\frac{1}{3}\epsilon} C_{\frac{1}{2}-\frac{1}{3}\epsilon} + C_{0,0,\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon,0,\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon} C_{0,\frac{1}{2}} \\
&\qquad + C_{\frac{1}{2}-\frac{1}{3}\epsilon} C_{\frac{1}{2}}(C_{\frac{1}{2},\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon,-\frac{1}{2},\frac{1}{2}} + \tau_5 C_{0,-\frac{1}{2},\frac{1}{2}})\Big] \\
&\qquad \cdot \max\left\{1, \frac{1}{T}\right\} \cdot \left(1 + \tau_5 T^{1/2}\right)^7 \cdot \left(1 + \frac{3T^{\frac{1}{2}+\frac{1}{3}\epsilon}}{\epsilon} + 2T\right)^3 \cdot K_5^5
\end{aligned}
\tag{3.81}
$$

where Corollary 3.5 shows that

$$
\begin{aligned}
K_5 \le 8&\Big[C_0 \max\{1, \|\xi_\delta\|_{L^5(\Omega,H)}\} + C_\vartheta \|F_\varepsilon\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} \\
&+ \tau_5 C_0 C_{G_\varepsilon}(1 + \tau_5 \|B_\varepsilon\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) T^{\frac{1}{2}}\Big]^2 \\
&\cdot \mathrm{E}_{1-\vartheta}\left(2\Gamma(2-\vartheta)\left(C_\vartheta |F_\varepsilon|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} + \tau_5 C_0 C_{G_\varepsilon}(1 + \tau_5 T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.82}
$$

This time we have to pay a price for the mollification. Because F is $H_{-\vartheta}$-valued and $\vartheta > \frac{1}{2} - \frac{1}{3}\epsilon$ we have to use the semigroup properties to bound

$$
\|F_\varepsilon\|_{C_b^3(H,H_{-\frac{1}{2}+\frac{1}{3}\epsilon})} \le C_{\vartheta-\frac{1}{2}+\frac{1}{3}\epsilon} \cdot \varepsilon^{-(\vartheta-\frac{1}{2}+\frac{1}{3}\epsilon)} \|F\|_{C_b^3(H,H_{-\vartheta})}
\tag{3.83}
$$

for all $\varepsilon \in (0,T]$. The diffusion term can as before be bounded as $\|B_\varepsilon\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} \le C_0 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}$.

Putting (3.79)–(3.83) together we arrive at

$$
\begin{aligned}
&\Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \\
&\quad \leq \Big(\varepsilon^{1-\vartheta-\epsilon} + \max\{1, \varepsilon^{-3(\vartheta-\frac{1}{2}+\frac{1}{3}\epsilon)}\}h^{\frac{1}{2}}\Big) \cdot 2304 \cdot C_0^{48} \cdot C^3_{\vartheta-\frac{1}{2}+\frac{1}{3}\epsilon} \\
&\qquad \cdot C_G^7 \cdot \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^3 \cdot \max\{1, \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\}^7 \\
&\qquad \cdot \max\left\{1, \frac{1}{T}\right\} \cdot \Big(1 + \tau_5 T^{1/2}\Big)^7 \cdot \left(1 + \frac{3T^{\frac{1}{2}+\frac{1}{3}\epsilon}}{\epsilon} + 2T\right)^3 \\
&\qquad \cdot \Big[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0,0} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0,0,0} \\
&\qquad + c^{(\varepsilon)}_{0,0} + c^{(\varepsilon)}_{0,0,0} + c^{(\varepsilon)}_{0,0,0,0} + \tilde{c}^{(\varepsilon)}_{0,0,0,0}\Big] \\
&\qquad \cdot \Big[C^2_{\frac{1}{2}} C_{1-\frac{1}{3}\epsilon} C_{\frac{1}{2}-\frac{1}{3}\epsilon} + C_{0,0,\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon,0,\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon} C_{0,\frac{1}{2}} \\
&\qquad + C_{\frac{1}{2}-\frac{1}{3}\epsilon} C_{\frac{1}{2}}(C_{\frac{1}{2},\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon,-\frac{1}{2},\frac{1}{2}} + \tau_5 C_{0,-\frac{1}{2},\frac{1}{2}})\Big] \\
&\qquad \cdot \Bigg[\max\{1, \|\xi\|_{L^5(\Omega,H)}\} + C_\vartheta C_{1-\vartheta-\epsilon} C_{1-\epsilon} \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{\epsilon} + 2\tau_5 C^2_{1-\vartheta-\epsilon} \\
&\qquad \cdot \max\{\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_5 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) \sqrt{\tfrac{T}{2\vartheta-1+2\epsilon}}\Bigg]^{10} \\
&\qquad \cdot \mathrm{E}_{1-\vartheta}\Bigg(2\Gamma(2-\vartheta)\Big(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} \\
&\qquad + \tau_5 C_0^2 \max\{|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_5 T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\Big)^2\Bigg)^{\frac{5}{2}}
\end{aligned}
\tag{3.84}
$$

which holds for all $\varepsilon, \delta \in (0, T]$. If we now couple the mollification parameter to the step size of the approximation as $\varepsilon = (h/T)^{\frac{1}{4\vartheta-1}}$ and remember that $\epsilon = 1 - \vartheta - (4\vartheta - 1)\rho$ then we can estimate the critical terms as

$$
\varepsilon^{1-\vartheta-\epsilon} + \max\{1, \varepsilon^{-3(\vartheta-\frac{1}{2}+\frac{1}{3}\epsilon)}\}h^{\frac{1}{2}} = \left(\frac{h}{T}\right)^{\frac{1-\vartheta-\epsilon}{4\vartheta-1}} + \left(\frac{h}{T}\right)^{-\frac{3(\vartheta-\frac{1}{2}+\frac{1}{3}\epsilon)}{4\vartheta-1}} h^{\frac{1}{2}} \tag{3.85}
$$

$$
\leq 2\max\{T, \frac{1}{T}\} h^{\frac{1-\vartheta-\epsilon}{4\vartheta-1}} \tag{3.86}
$$

$$
= 2\max\{T, \frac{1}{T}\} h^{\rho}. \tag{3.87}
$$

Next we use again Theorem 1.38 to see that the constants arising from the Kolmogorov backward equation are bounded for $\varepsilon \in (0, T]$ and obtain for all $\delta \in (0, T]$

$$
\begin{aligned}
&\Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \\
&\quad \leq h^\rho \cdot 4608 \cdot C_0^{48} \cdot C^3_{\vartheta-\frac{1}{2}+\frac{1}{3}\epsilon} \cdot C_G^7 \cdot \max\{1, \|F\|_{C_b^3(H,H_{-\vartheta})}\}^3 \\
&\qquad \cdot \max\{1, \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}\}^7 \\
&\qquad \cdot \max\left\{T, \frac{1}{T}\right\}^2 \cdot \left(1 + \tau_5 T^{1/2}\right)^7 \cdot \left(1 + \frac{3T^{\frac{1}{2}+\frac{1}{3}\epsilon}}{\epsilon} + 2T\right)^3 \\
&\qquad \cdot \Bigg[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + \sup_{\varepsilon\in(0,T]} \Big[c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0,0} + c^{(\varepsilon)}_{\frac{1}{2}-\frac{1}{3}\epsilon,0,0,0} \\
&\qquad + c^{(\varepsilon)}_{0,0} + c^{(\varepsilon)}_{0,0,0} + c^{(\varepsilon)}_{0,0,0,0} + \tilde{c}^{(\varepsilon)}_{0,0,0,0}\Big]\Bigg] \\
&\qquad \cdot \Big[C^2_{\frac{1}{2}} C_{1-\frac{1}{3}\epsilon} C_{\frac{1}{2}-\frac{1}{3}\epsilon} + C_{0,0,\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon,0,\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon} C_{0,\frac{1}{2}} \\
&\qquad + C_{\frac{1}{2}-\frac{1}{3}\epsilon} C_{\frac{1}{2}} (C_{\frac{1}{2},\frac{1}{2}} + C_{\frac{1}{2}-\frac{1}{3}\epsilon,-\frac{1}{2},\frac{1}{2}} + \tau_5 C_{0,-\frac{1}{2},\frac{1}{2}})\Big] \\
&\qquad \cdot \Bigg[\max\{1, \|\xi\|_{L^5(\Omega,H)}\} + C_\vartheta C_{1-\vartheta-\epsilon} C_{1-\epsilon} \|F\|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{\epsilon} + 2\tau_5 C^2_{1-\vartheta-\epsilon} \\
&\qquad \cdot \max\{\|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_5 \|B\|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))} T^{\frac{1}{2}}) \sqrt{\tfrac{T}{2\vartheta-1+2\epsilon}}\Bigg]^{10} \\
&\qquad \cdot \mathrm{E}_{1-\vartheta}\Bigg(2\Gamma(2-\vartheta)\Big(C_0 C_\vartheta |F|_{\mathrm{Lip}(H,H_{-\vartheta})} \tfrac{T^{1-\vartheta}}{1-\vartheta} \\
&\qquad + \tau_5 C_0^2 \max\{|B|_{\mathrm{Lip}(H,\mathrm{L}_2(U,H))}, C_G\}(1 + \tau_5 T^{\frac{1}{2}}) T \sqrt{\tfrac{T^{1-\vartheta}}{1-\vartheta}}\Big)^2\Bigg)^{\frac{5}{2}}.
\end{aligned} \tag{3.88}
$$

Because the right-hand side of (3.88) does not depend on δ and since

$$
\begin{aligned}
&\Big\|\mathbb{E}\Big[\varphi(Z_T) - \varphi(Z_T^{0,\delta})\Big]\Big\|_V + \Big\|\mathbb{E}\Big[\varphi(X_T^{0,\delta}) - \varphi(X_T)\Big]\Big\|_V \\
&\quad \leq |\varphi|_{\mathrm{Lip}(H,V)} \Big\|Z_T - Z_T^{0,\delta}\Big\|_{L^2(\Omega,H)} + |\varphi|_{\mathrm{Lip}(H,V)} \Big\|X_T^{0,\delta} - X_T\Big\|_{L^2(\Omega,H)}
\end{aligned} \tag{3.89}
$$

we can use Lemmas 3.8 and 3.9 together with (3.88) to get that

$$
\begin{aligned}
\|\mathbb{E}[\varphi(Z_T) - \varphi(X_T)]\|_V &\leq \lim_{\delta\to 0} \Big\|\mathbb{E}\Big[\varphi(Z_T) - \varphi(Z_T^{0,\delta})\Big]\Big\|_V \\
&\quad + \Big\|\mathbb{E}\Big[\varphi(Z_T^{0,\delta}) - \varphi(X_T^{0,\delta})\Big]\Big\|_V \qquad (3.90) \\
&\quad + \Big\|\mathbb{E}\Big[\varphi(X_T^{0,\delta}) - \varphi(X_T)\Big]\Big\|_V \\
&\leq C \cdot h^\rho \qquad (3.91)
\end{aligned}
$$

where $C \in [0,\infty)$ is the constant on the right-hand side of (3.88). $\square$

Conclusion

We have proved new convergence rates of time-discrete schemes for the weak approximation of the mild solution of semi-linear parabolic stochastic evolution equations. The rates of convergence depend on the overall regularity parameter $\vartheta := \max\{\alpha, 2\beta\}$ which is composed of the regularity $\alpha \in [0, 1)$ of the drift coefficient and the regularity $\beta \in [0, 1/2)$ of the diffusion coefficient. In the setting of this thesis, only regular diffusion coefficients ($\beta = 0$) are allowed for the Milstein-type schemes and thus $\vartheta = \alpha$. In particular, we have shown that the considered Milstein-type schemes achieve a weak convergence rate of almost

$$\rho^{\mathrm{M}}(\vartheta) := \begin{cases} 1 - \vartheta & \vartheta \in [0, \frac{1}{2}), \\ \frac{1-\vartheta}{4\vartheta-1} & \vartheta \in [\frac{1}{2}, 1), \end{cases}$$

i.e. a rate of $\rho^{\mathrm{M}} - \varepsilon$ for arbitrarily small $\varepsilon > 0$ in the number of time steps.

Let us compare this to [JK21]. The authors investigated Euler-type schemes and proved a weak convergence rate of almost

$$\rho^{\mathrm{E}}(\vartheta) := \begin{cases} 1 - \vartheta & \vartheta \in [0, \frac{1}{2}), \\ \frac{1-\vartheta}{10\vartheta-4} & \vartheta \in [\frac{1}{2}, 1). \end{cases}$$

Note that the restriction to $\beta = 0$ is not necessary for Euler-type schemes. Both convergence rates are visualised in Figure 3.1. We can see that the rates coincide for small ϑ. In the regime $\vartheta > \frac{1}{2}$, we have $\rho^{\mathrm{M}} > \rho^{\mathrm{E}}$, which seems to suggest an improved convergence rate of the Milstein-type schemes over the Euler-type schemes. However, this small improvement is solely an artefact of the restriction $\beta = 0$ that we must impose in order for the Milstein-type schemes to be well-defined. If we repeated the proof in [JK21] under the assumption that $\beta = 0$, we would obtain the same improved rate ρ^{M} for the Euler-type schemes as well. The remaining advantage of the Milstein-type schemes would thus be an improved strong convergence rate. But we emphasise again that in our setting no proof of a strong convergence rate is currently known. Knowledge of both strong and weak rates would allow applications to multilevel Monte Carlo schemes, see for instance [Gil08; Lan16].

The form of the proved convergence rates for Euler- and Milstein-type schemes very strongly suggests that in both cases $1 - \vartheta$ should be the correct order for all $\vartheta \in [0, 1)$. Nevertheless, without an improved result on the regularity of solutions of infinite-dimensional Kolmogorov backward equations (compare Theorem 1.37) we are not able to prove this conjecture. In particular, we would need to allow $\delta_i \in [0, 1)$ in Theorem 1.37 instead of $\delta_i \in [0, \frac{1}{2})$ and also lift the restriction $\sum_{i=1}^{k} \delta_i < \frac{1}{2}$ to at least allow $\sum_{i=1}^{k} \delta_i < 1$. See also the introduction in [BD18] for a detailed account of this problem.

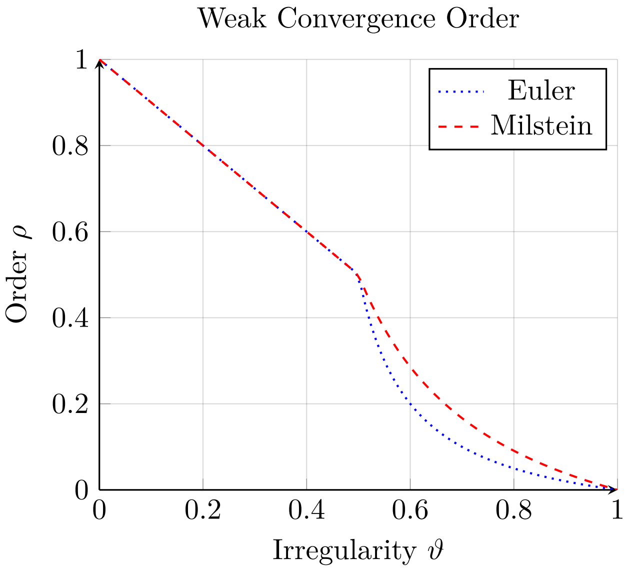

Figure 3.1: Convergence orders for the Euler-type schemes from [JK21] (blue, dotted) and the Milstein-type schemes considered here (red, dashed) depending on the regularity of the drift $F\colon H \to H_{-\vartheta}$ and diffusion $B\colon H \to \mathrm{L}_2(U, H_{-\vartheta/2})$. For the Milstein-type schemes we require $B\colon H \to \mathrm{L}_2(U, H)$.

In this thesis, we focused only on the time-discrete approximation. Since the Hilbert spaces on which the solution lives are in general infinite-dimensional, we have to combine the obtained results with a discretisation in space. For this purpose, we refer to [CJK19] as an example of spatial discretisation by Galerkin methods. In addition, the Wiener process also lives on a possibly infinite-dimensional Hilbert space. In order to successfully implement such a numerical scheme we therefore have to discretise the noise as well. The weak convergence of the noise discretisation was for instance investigated in [HM19]. The combination of these three building blocks was not covered in this thesis and is the subject of future work.

Another possibility for further research directions is provided by Milstein-type schemes that are not covered by our setting, for example, more efficient derivative-free Milstein schemes [HR23] or so-called tamed Milstein schemes [HJK12; WG13b] as well as an extension to Banach space-valued processes [HJK16].

Lastly, we aim to investigate the feasibility of lifting the $\beta = 0$ restriction of the Milstein scheme by incorporating a small modification into the crucial iterated integral term.

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